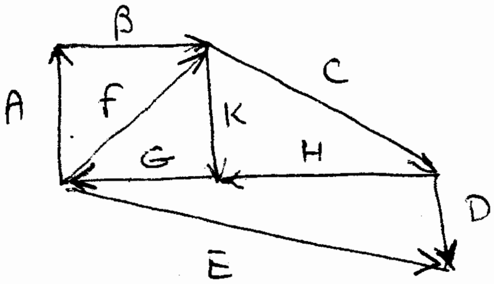
MA 1116 — Suggested Homework Problems from Davis & Snider 7^{th} edition

Sec.	Page	Problems
	Chapter 1	
1.2	pg. 6	1–4, 8
1.3	pg. 8	2, 5, 8, 14
1.4	pg. 10	4, 6, 9, 12
1.5	pg. 14	4–6, 13, 15
1.6	pg. 17	3, 5
1.7	pg. 23	1, 3, 7, 9, 17, 19
1.8	pg. 29	1, 4, 7, 9, 11, 15a
1.9	pg. 34	1, 5, 7, 11, 12a, 21, 24
1.10	pg. 38	2, 4, 6, 9, 12, 23
1.12	pg. 51	1, 3, 5, 8, 10, 13, 19, 23
1.13	pg. 57	1, 3, 5, 7
1.14	pg. 60	5–7
	Chapter 2	
2.1	pg. 70	1, 3, 4, 5hi
2.2	pg. 85	1–3, 5, 8, 13
2.3	pg. 95	2, 5–7, 9, 15, 17
2.4	pg. 102	1, 3, 4, 6, 10
	Chapter 3	
3.1	pg. 112	1, 6, 7, 8, 16, 17, 21, 24, 31
3.2	pg. 117	1–4
3.3	pg. 124	1, 3, 5, 7, 10
3.4	pg. 132	1, 2, 4, 7, 10, 12
3.5	pg. 135	1, 2, 4–8
3.6	pg. 140	1, 3–5, 7
3.8	pg. 150	1, 5, 6, 10, 12
3.10	pg. 169	2, 4, 6, 8, 11
3.11	pg. 180	3–5, 8, 9, 12, 13
	Chapter 4	
4.1	pg. 190	1, 3, 6, 10, 14, 18
4.3	pg. 204	2, 4–7
4.4	pg. 212	1, 2, 6, 7, 9, 10
4.6	pg. 236	1, 3–6
4.7	pg. 246	1, 2d, 5, 6, 10, 15, 19
4.8	pg. 256	2–4, 6
4.9	pg. 262	3, 4, 6, 9, 14, 17
4.10	pg. 271	1
	Chapter 5	
5.1	pg. 276	6–9
5.2	pg. 284	1, 2, 5, 8a
5.4	pg. 294	4, 5, 7, 9, 12
5.5	pg. 299	1–3, 7

1.2 p. 6

Fig. 1.6



(1.) C in terms of E, D, F

$$F + C + D = E$$

$$C = E - D - F$$

(2.) G in terms of C, D, E, K

$$K + G = -F$$

$$F + C + D = E$$

$$K + G = -E + C + D$$

$$\underline{G = -E + C + D - K}$$

(3)

$$x + B = F$$

$$x = F - B = A \quad \text{(figure 1.6)}$$

(4)

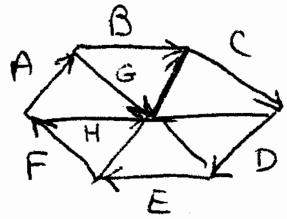
$$x + H = D - E$$

$$D - E = G + H \quad \text{(figure 1.6)}$$

$$\Rightarrow x = G$$

1.2 p.6

(8)



regular means six equilateral triangles

(a) $C = B - A$ ($G = C$ but $G = B - A$ since $H = B$

$D = -A$

$E = -B$

$F = -C = A - B$

(b) sum of all is zero (vector addition last terminal point is the same as first initial point.)

1.3 p. 8

2. If $|A| = 3$

$|4A| = 4|A| = 4 \cdot 3 = 12$

$|-2A| = |-2| \cdot |A| = 2 \cdot 3 = 6$

$|sA|$; $-2 \le s \le 1$ $\qquad |sA| = |s| \cdot |A| = 3|s| \le 3 \cdot 2 = 6$

(5) $A = \alpha B$

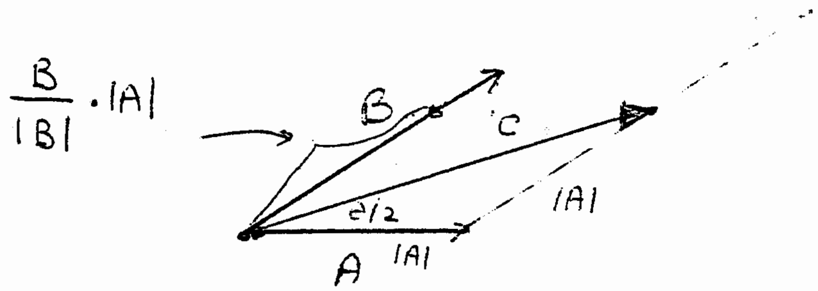
Is $B \stackrel{?}{=} \gamma A$ (yes if $\alpha \neq 0$, we can divide by $\alpha \Rightarrow$

$B = \underbrace{\frac{1}{\alpha}}_{=\gamma} A$

If $\alpha = 0$ then $|A| = 0$ and one cannot have any vector (except zero) as a multiple of zero}.

(8) Two vectors, one pointing up from plane and one downward

14.



equilateral triangle

$$C = A + \underbrace{B\,\frac{|A|}{|B|}}$$

vector of length |A| in the direction of B so that the angle is bisected.

1.4 p.10

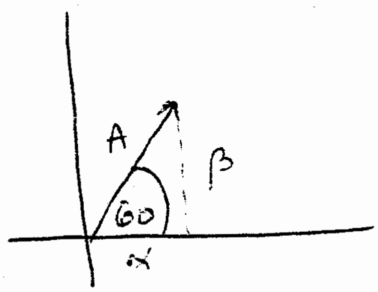
(4) $|3i-4j| = \sqrt{3^2+(-4)^2} = \sqrt{9+16} = 5$

(6) $P_1(4,2) \quad P_2(5,-1)$

$\overrightarrow{P_1P_2} = (5-4)\,i + (-1-2)\,j$

$= \underline{i-3j}$

(9) (a)



$A = \alpha i + \beta j$

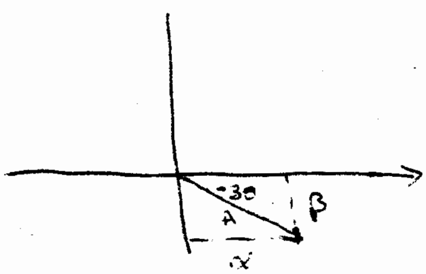
$|A| = \sqrt{\alpha^2+\beta^2} = 1$ unit vector

$\underbrace{\sin 60^\circ}_{1/2} = \dfrac{\beta}{|A|} = \beta \Rightarrow \beta = 1/2$

$\alpha = \sqrt{1-\beta^2} = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2}$

$\boxed{A = \frac{\sqrt{3}}{2}i + \frac{1}{2}j}$

(b)



$\dfrac{\beta}{|A|} = \sin(-30) = -\frac{\sqrt{3}}{2}$

$\beta = -\frac{\sqrt{3}}{2}$

$\alpha = 1/2$

$A = \frac{1}{2}i - \frac{\sqrt{3}}{2}j$

1.4. p.10

9 c $|3i+4j| = 5$

$\frac{3}{5}i + \frac{4}{5}j$ is in the same direction as original

also it is of unit length (dividing by original length)

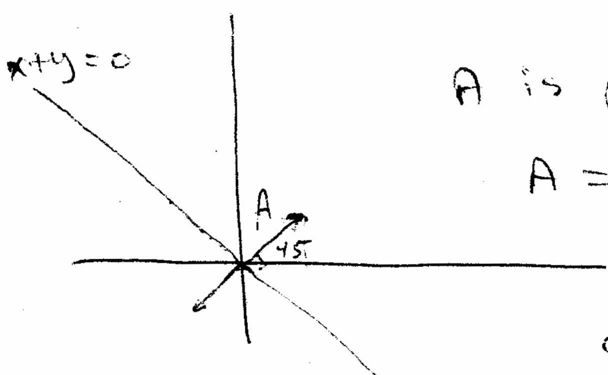
d $A = \frac{1}{2}i + \beta j$

$\beta^2 + \left(\frac{1}{2}\right)^2 = 1 \Rightarrow \beta^2 = 1 - \frac{1}{4} = \frac{3}{4}$

$\beta = \pm\frac{\sqrt{3}}{2}$

$\underline{A = \frac{1}{2}i \pm \frac{\sqrt{3}}{2}j}$

e



A is perpendicular to line

$A = \alpha i + \beta j$

$\alpha^2 + \beta^2 = 1$ (looking for unit vector)

$\alpha = \beta = \pm\frac{\sqrt{2}}{2}$ see figure

$A = \pm\frac{\sqrt{2}}{2}(i + j)$

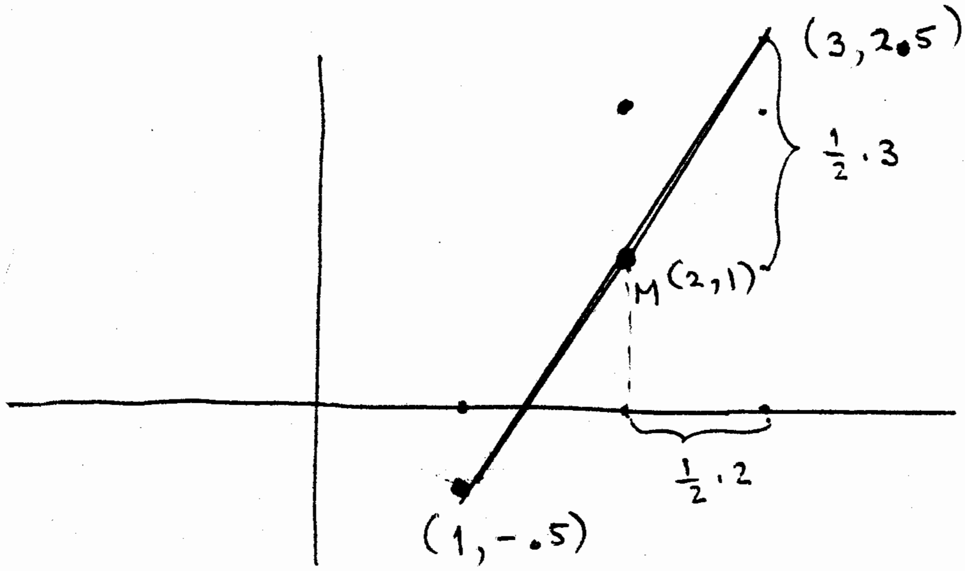
Only two vectors!!

1.4 p.10

12 $V = 2i + 3j$

mid pt: M (2, 1)

draw:



$A = (1, -.5)$

$B = (3, 2.5)$

1.5 p.14

(4) $B = 2i + 2j - k \qquad |B| = \sqrt{4+4+1} = 3$

$|sB| = 1$

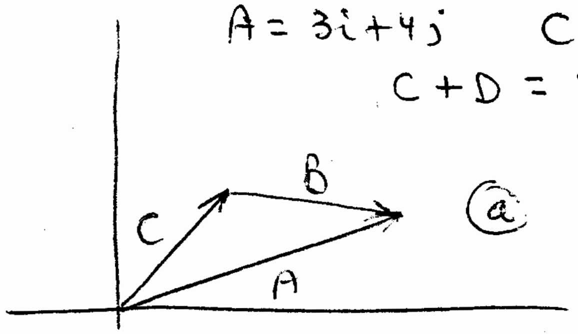
$|s| = \dfrac{1}{|B|} \qquad s = \pm \dfrac{1}{|B|} = \pm \dfrac{1}{3}$

(5) $A = 3i + 4j \qquad |A| = \sqrt{9+16} = 5$

$\dfrac{A}{|A|} \qquad \dfrac{3}{5} i + \dfrac{4}{5} j$

(6) $A = 3i + 4j \qquad C = 3i - 4k \Rightarrow D = 4j + 4k$

$C + D = A \Rightarrow D = A - C$



(a) $|D| = |A - C| = \sqrt{16+16} = 4\sqrt{2}$

(b) yz plane (no compon. in x direction)

(13) $A = 2i - 2j + k$

$|A| = \sqrt{4+4+1} = 3$

$\left(\dfrac{2}{3}, -\dfrac{2}{3}, \dfrac{1}{3}\right)$

(15) $\cos x = 1/2$

cone vertex at O

base perpend. to x axis

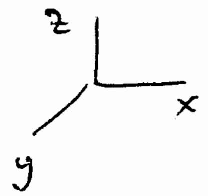
1.6 p.17

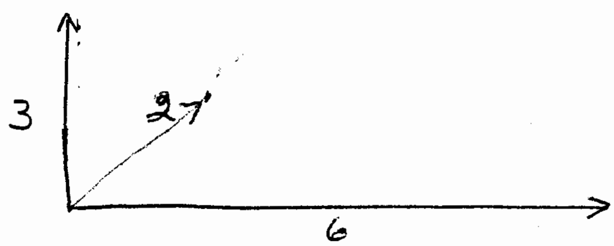
(3)

$$R_1(t=1) = 3i + 4j - k$$
$$R_2(t=3) = 3i + 36j - 27k$$

$$\text{displacement} = R_2 - R_1 = 32j - 26k$$

(5)





$$F_1 = 3k$$
$$F_2 = 6i$$
$$F_3 = -2j$$
$$F_4 = ?$$

$$F_1 + F_2 + F_3 + F_4 = 0$$

$$F_4 = -F_1 - F_2 - F_3 = \underline{-6i + 2j - 3k}$$

$$|F_4| = \sqrt{36+4+9} = \sqrt{49} = \underline{\underline{7\ lb}}$$

1.7 p.23

(1.) $A = 2i + j + 2k$ $|A| = \sqrt{4+1+4} = 3$

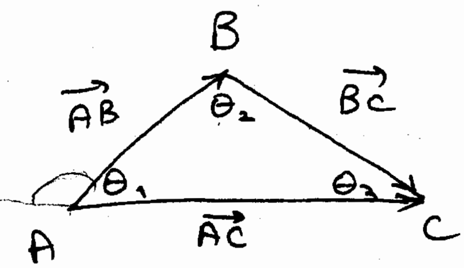
$B = 3i - 4k$ $|B| = \sqrt{9+16} = 5$

$$\cos\theta = \frac{6+0-8}{3\cdot 5} = \frac{-2}{15} \qquad \theta = \arccos\left(-\frac{2}{15}\right)$$

(3) Find the three angles of the triangle with vertices $A = (2,-1,1)$

$B = (1,-3,-5)$

$C = (3,-4,-4)$



Sides: $\vec{AB} = -i - 2j - 6k$

$\vec{AC} = i - 3j - 5k$

$\vec{BC} = 2i - j + k$

$$\cos\theta_1 = \frac{\vec{AB}\cdot\vec{AC}}{|\vec{AB}|\cdot|\vec{AC}|} = \frac{-1+6+30}{\sqrt{1+4+36}\sqrt{1+9+25}} = \frac{\cancel{35}^{\sqrt{35}}}{\sqrt{41}\,\cancel{\sqrt{35}}} = \frac{\sqrt{1435}}{41}$$

$$\theta_1 = \arccos\left(\frac{\sqrt{1435}}{41}\right)$$

since I need vectors pointing away from B.

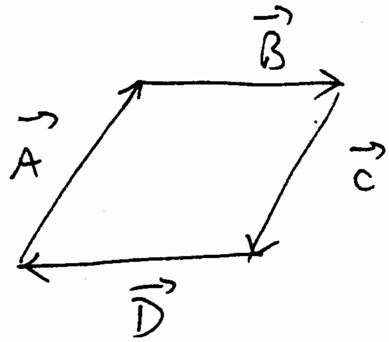
$$\cos\theta_2 = \frac{-\vec{AB}\cdot\vec{BC}}{|AB|\cdot|\vec{BC}|} = \frac{-(-2+2-6)}{\sqrt{41}\sqrt{4+1+1}} = \frac{\cancel{6}^{\sqrt{6}}}{\sqrt{41}\,\cancel{\sqrt{6}}} = \frac{\sqrt{246}}{41}$$

$$\theta_2 = \arccos\frac{\sqrt{246}}{41}$$

$$\theta_3 = 180 - \theta_1 - \theta_2$$

1.7 p. 23 cont.

(7)



$\vec{A}+\vec{B}+\vec{C}+\vec{D}=0 \qquad (1)$

$\vec{A}=-\vec{C}$ (parallel & equal)

$\Downarrow$ substitute in (1)

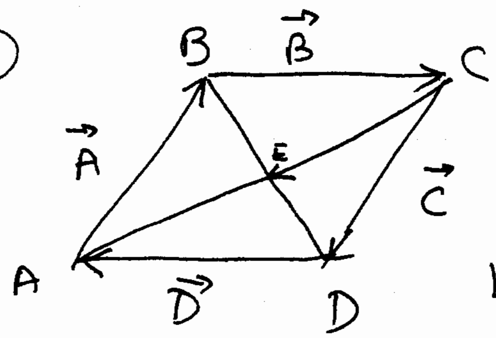
$\vec{B}+\vec{D}=0$

$\Downarrow$

$\vec{B}=-\vec{D}$

other two are parallel & equal

(9)



show $AE = EC$

& $BE = ED$

1. $\vec{A}+\vec{B}+\vec{C}+\vec{D}=0$
2. $\vec{CA}=\vec{A}+\vec{B}$

Let's look at $\vec{CE}$ and prove

$\vec{CE}=\frac{1}{2}\vec{CA}=\frac{1}{2}(\vec{A}+\vec{B})$

[similarly one can prove $\vec{BE}=\frac{1}{2}\vec{BD}$]

$\vec{CE}=s(\vec{A}+\vec{B})$ since it is in the direction of $\vec{A}+\vec{B}$

E is on the line connecting B & D $\Rightarrow$

$-\vec{EC}=\vec{C}+t(\vec{B}+\vec{C})$ but $\vec{C}=-\vec{A}$

$=-\vec{A}+t(\vec{B}-\vec{A})$

$\Rightarrow \vec{EC}=\vec{A}+t(\vec{A}-\vec{B}) = s(\vec{A}+\vec{B})$

$\Rightarrow (s-1-t)\vec{A}=(-s-t)\vec{B}$

since $\vec{A}$, $\vec{B}$ are not parallel the scalars are zero

$$\left.\begin{array}{l} s-t-1=0 \\ s+t=0 \end{array}\right\} +$$

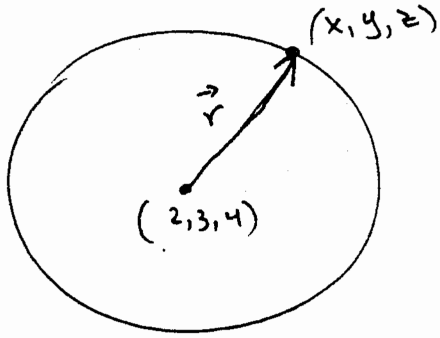
$$2s-1=0$$

$$s=\tfrac{1}{2}$$

$$\Rightarrow \vec{EC} = s(\vec{A}+\vec{B}) = \tfrac{1}{2}(\vec{A}+\vec{B}) \quad \square$$

1.7 p.23 cont.

(17)



$$\rho = |r| = (x-2)^2 + (y-3)^2 + (z-4)^2$$

(19)

$x = y = z$ line or a plane?

line!

$x = y$ is a plane

$x = z$ is a plane

and we are taking the intersection of those

1.8 p.29

(1) line thru $(0,0,0)$ parallel $3i - 2j + 7k$

$$\frac{x}{3} = \frac{y}{-2} = \frac{z}{7}$$

or $\begin{cases} x = 3t \\ y = -2t \\ z = 7t \end{cases}$ parametric

(4) Find the two unit vectors parallel to the line

$$\frac{x-1}{3} = \frac{y+2}{4} \qquad z = 9$$

$$v = 3i + 4j$$

$$|v| = \sqrt{9+16} = \pm 5$$

$$\frac{v}{|v|} = \pm \frac{3}{5} i \pm \frac{4}{5} j$$

(7) Find equations of line thru $(0,0,0)$ parallel to

$$x - 3 = \frac{y+2}{4} = -z+1$$

$$v = i + 4j - k$$

$$\frac{x}{1} = \frac{y}{4} = \frac{z}{-1}$$

1.8 p. 29

(9) line thru $(1, 4, -1)$ $(2, 2, 7)$

$$v = (2-1)\,i + (2-4)\,j + (7+1)\,k$$
$$v = i - 2j + 8k$$

$$\frac{x-1}{1} = \frac{y-4}{-2} = \frac{z+1}{8}$$

or

$$\frac{x-2}{1} = \frac{y-2}{-2} = \frac{z-7}{8}$$

(11) angle between

$$\frac{x-1}{3} = \frac{y-3}{4} = \frac{z}{5}$$

$$\frac{x-1}{2} = 3-y = 2z$$

$$v = (3, 4, 5) \qquad |v| = \sqrt{9+16+25} = \sqrt{50}$$

$$w = (2, -1, \tfrac{1}{2}) \qquad |w| = \sqrt{4+1+\tfrac{1}{4}} = \sqrt{\tfrac{21}{4}}$$

$$\cos\theta = \frac{3\cdot 2 + 4(-1) + 5\cdot\frac{1}{2}}{\sqrt{50}\,\sqrt{\frac{21}{4}}} = \frac{6-4+\frac{5}{2}}{\sqrt{50}\,\frac{\sqrt{21}}{2}} = \frac{9/\cancel{2}}{\sqrt{50}\cdot\sqrt{21}/\cancel{2}}$$

$$\cos\theta = \frac{9}{5\sqrt{21}\,\sqrt{2}}$$

1.8 p. 29

15a points of intersection

$$\vec{R} = (5i + 4j + 5k)t + 7i + 6j + 8k$$

$$\vec{R} = (6i + 4j + 6k)t + 8i + 6j + 9k$$

$$\frac{x-7}{5} = \frac{y-6}{4} = \frac{z-8}{5}$$

$$\frac{x-8}{6} = \frac{y-6}{4} = \frac{z-9}{6}$$

$$\frac{x-7}{5} = \frac{y-6}{4} = \frac{z-8}{5}; \qquad \frac{x-8}{6} = \frac{y-6}{4}$$

$$\frac{x-7}{5} = \frac{x-8}{6} \Rightarrow 6x - 42 = 5x - 40$$

$$x = 2$$

$$\frac{x-7}{5} = \frac{y-6}{4}$$

$$-\frac{5}{5} = \frac{y-6}{4} \qquad y - 6 = -4$$

$$y = 2$$

$$\frac{x-7}{5} = \frac{z-8}{5} \Rightarrow -1 = \frac{z-8}{5} \qquad z - 8 = -5$$

$$z = 3$$

Does this satisfy the last piece

$$\frac{y-6}{4} = \frac{z-9}{6}$$

$$y = 2 \qquad z = 3$$

$$-1 \stackrel{?}{=} \frac{-6}{6}$$

yes! $(2, 2, 3)$

1.9 p.34

(1) $(3i+8j-2k)\cdot(5i+j+2k) = 3\cdot5+8\cdot1-2\cdot2 = 15+8-4 = \underline{19}$

(5) $2i \qquad 3i+4j$

$$\cos\theta = \frac{2\cdot3}{\sqrt{4}\sqrt{9+16}} = \frac{6}{2\cdot5} = .6 \qquad \theta = 53.13^\circ$$

(7) $\underset{B}{8i+j}$ in the direction of $\underset{A}{i+2j-2k}$

$$\frac{A\cdot B}{|A|} = \frac{(8i+j)\cdot(i+2j-2k)}{\sqrt{1+4+4}} = \frac{8+2}{\sqrt{9}} = \frac{10}{3}$$

(11) $\left.\begin{array}{l} A\cdot A=0 \\ \Downarrow \\ \vec{A}=\vec{0} \end{array} \quad A\cdot B=0 \right\} \Rightarrow$ no conclusion on B.

(12) $B = 6i-3j-6k$

(a) $A = i+j+k$

$$B_{\parallel} = \frac{B\cdot A}{A\cdot A}A = \frac{6-3-6}{1+1+1}(i+j+k) = -i-j-k$$

$$B_{\perp} = 6i-3j-6k-(-i-j-k) = 7i-2j-5k$$

(21) $|A+B| \leq |A| + |B|$

square LHS $|A+B|^2 = (A+B)\cdot(A+B) = A\cdot A + A\cdot B + \underbrace{B\cdot A}_{A\cdot B} + B\cdot B$

$= |A|^2 + 2A\cdot B + |B|^2$

square RHS $(|A|+|B|)^2 = |A|^2 + 2|A||B| + |B|^2$

we need to show that $2A\cdot B \leq 2|A||B|$

Now $A\cdot B = |A||B|\cos\theta$

Since $|\cos\theta| \leq 1$

we have the triangle inequality.

(24) l_1 thru $(5,1,-2)$ $(2,-3,1)$

l_2 thru $(3,8,1)$ $(-3,0,7)$

$v_1 = 3i + 4j - 3k$ along l_1

$v_2 = 6i + 8j - 6k$ along l_2

$v_1 \cdot v_2 = 3\cdot 6 + 4\cdot 8 + (-3)(-6) \neq 0$ not perpedicular

Note : $v_2 = 2v_1 \Rightarrow$ lines are parallel

1.10 p.38

(2) plane thru (0,0,0) $\perp$ $2i-8j+2k$

$2x-8y+2z=0$ can divide thru by 2.

(4) plane parallel to $3x+y-z=8$ thru (1,3,3)

Parallel planes having same normal $3i+j-k$

$3x+y-z=3\cdot 1\cancel{+3-3}=3$

$3x+y-z=3$

(6)

$\vec{R}_1=3i+4j+7k$ $\quad d=4$

$\vec{n}=2i-j-2k$

$$\text{distance}=\frac{|R_1\cdot n-d|}{|n|}=\frac{|6-4-14-4|}{\sqrt{4+1+4}}=\frac{16}{3}$$

(9) $x=y=\frac{z+2}{3} \parallel 2x-8y+2z=5$

$\Rightarrow$ show that the normal to plane $\perp$ line

$n=2i-8j+2k$ vector normal to plane

$v=i+j+3k$ vector parallel line

$n\cdot v=2-8+6=0$ 0

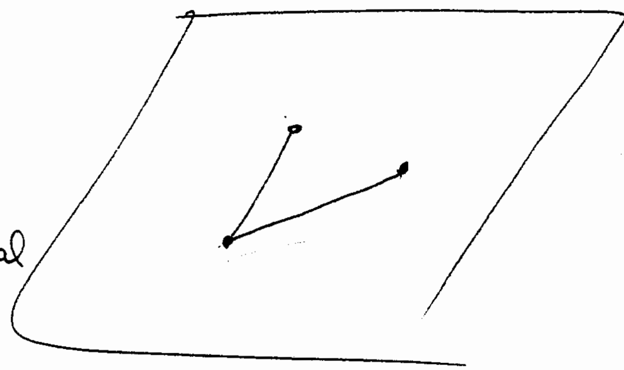
1.10 p.38

(12) $O(0,0,0)$ $\quad A(1,2,3)$ $\quad B(0,-1,1)$

(a)

$\vec{OA} = i + 2j + 3k$

$\vec{OB} = -j + k$



$OA \times OB$ = vector normal to plane.

Since we didn't learn this we can use

$ax+by+cz=d$ Find a,b,c,d so that the 3 points are in the plane.

Using O: $\underline{0 = d}$

A: $a + 2b + 3c = 0$

B $-b + c = 0 \Rightarrow \boxed{b = c}$

$a + 2b + 3b = 0$

$a + 5b = 0$

$\underline{a = -5b}$

We have a choice. $\underline{\text{For } b = 1} \Rightarrow c = 1, \; a = -5$

$\boxed{-5x + y + z = 0}$ Notice a different choice of b will give a multiple of this eq.

$\Rightarrow \quad n = -5i + j + k$

(b) distance from $C(2,0,2)$

$$\frac{|-5 \cdot 2 + 1 \cdot 2 - 0|}{\sqrt{25+1+1}} = \frac{|-10+2|}{\sqrt{27}} = \underline{\underline{\frac{8}{\sqrt{27}}}}$$

1.10 p. 38

23 plane thru $(4,0,0)$ $(0,6,0)$ $(0,0,12)$

Find equation of another plane thru $(6,-2,4)$ $\parallel$ to this.

$$ax + by + cz = d$$
$$4a = d$$
$$6b = d$$
$$12c = d$$

choose $\underline{d = 12}$ $\Rightarrow$ $c = 1$, $b = 2$, $a = 3$

$$\boxed{3x + 2y + z = 12}$$

$n = 3i + 2j + k$

The other plane has same normal

$3x + 2y + z = d$ thru $(6,-2,4)$

$d = 3 \cdot 6 - 2 \cdot 2 + 4 = 18 - 4 + 4 = 18$

$$\boxed{3x + 2y + z = 18}$$

1.12 p. 51

(1) a $A = 3i - j + 2k$

$B = i + j - 4k$

$$A\times B = \begin{vmatrix} i & j & k \\ 3 & -1 & 2 \\ 1 & 1 & -4 \end{vmatrix} = i\begin{vmatrix} -1 & 2 \\ 1 & -4 \end{vmatrix} - j\begin{vmatrix} 3 & 2 \\ 1 & -4 \end{vmatrix} + k\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= i(4-2) - j(-12-2) + k(3+1)$$

$$= \underline{2i + 14j + 4k}$$

b $$A\times B = \begin{vmatrix} i & j & k \\ 2 & 1 & 7 \\ 3 & 1 & -1 \end{vmatrix} = i\begin{vmatrix} 1 & 7 \\ 1 & -1 \end{vmatrix} - j\begin{vmatrix} 2 & 7 \\ 3 & -1 \end{vmatrix} + k\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}$$

$$= i(-8) - j(-2-21) + k(2-3)$$

$$= \underline{-8i + 23j - k}$$

c $$A\times B = \begin{vmatrix} i & j & k \\ 0 & 1 & 6 \\ -1 & 2 & 1 \end{vmatrix} = i\begin{vmatrix} 1 & 6 \\ 2 & 1 \end{vmatrix} - j\begin{vmatrix} 0 & 6 \\ -1 & 1 \end{vmatrix} + k\begin{vmatrix} 0 & 1 \\ -1 & 2 \end{vmatrix}$$

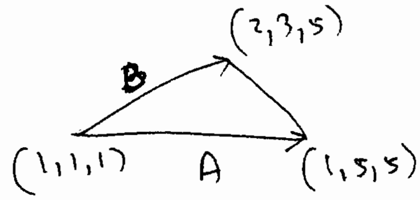
$$= \underline{-11i - 6j + k}$$

d $$A\times B = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = i\underbrace{\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}}_{=0} - j\underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix}}_{=0} + k\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= \underline{k}$$

e $B\times A = i - j \Rightarrow A\times B = -i + j$

3 area of triangle



$B = i + 2j + 4k$

$A = \quad 4j + 4k$

$$\text{area} = \frac{1}{2}\left|(A \times B)\right|$$

$$A \times B = \begin{vmatrix} i & j & k \\ 0 & 4 & 4 \\ 1 & 2 & +4 \end{vmatrix} = i\begin{vmatrix} 4 & 4 \\ 2 & +4 \end{vmatrix} - j\begin{vmatrix} 0 & 4 \\ 1 & +4 \end{vmatrix} + k\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix}$$

$$= i(+16-8) - j(-4) + k(-4)$$

$$= +8i + 4j - 4k = 4(2i + j - k)$$

$$|A \times B| = \sqrt{8^2+4^2+4^2} = \sqrt{64+16+16} = \sqrt{96} = 4\sqrt{6}$$

$$\text{area} = 2\sqrt{6}$$

5 $\quad A = 3i + j \qquad B = 2i - j - 5k$

$$A \times B = \begin{vmatrix} i & j & k \\ 3 & 1 & 0 \\ 2 & -1 & -5 \end{vmatrix} = i(-5) - j(-15) + k(-5)$$

$$= -5i + 15j - 5k = -5(i - 3j + k)$$

$$|A \times B| = \sqrt{1+9+1} = \sqrt{11}$$

$$\frac{A \times B}{|A \times B|} = \frac{i - 3j + k}{\sqrt{11}}$$

(8) $A = 2i + 2j \quad B = 3i - j + k \quad C = 8i$

$$A \times B = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 3 & -1 & 1 \end{vmatrix} = 2i - 2j - 8k$$

$$B \times C = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ 8 & 0 & 0 \end{vmatrix} = 8j + 8k$$

$$(A \times B) \times C = \begin{vmatrix} i & j & k \\ 2 & -2 & -8 \\ 8 & 0 & 0 \end{vmatrix} = -64j + 16k$$

$$A \times (B \times C) = \begin{vmatrix} i & j & k \\ 2 & 2 & 0 \\ 0 & 8 & 8 \end{vmatrix} = 16i - 16j + 16k$$

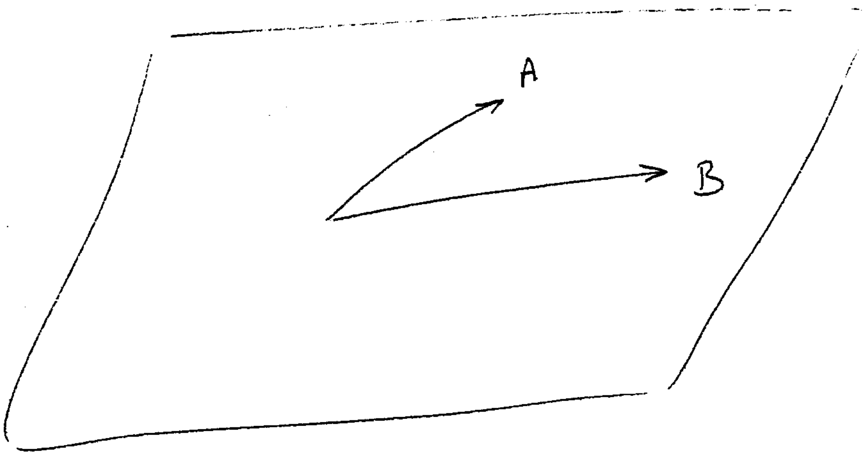
NO

1.12 p.51 cont.

(10) Find a unit vector in the plane of $A = i + 2j$
$B = j + 2k$

perp. to $C = 2i + j + 2k$

$A \times B$ is perpendicular to A, B $= \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 4i - 2j + k$



$(A \times B) \times C$ is perp to $A \times B$, C

$\Rightarrow$ perp to C and in plane of A, B

$\begin{vmatrix} i & j & k \\ 4 & -2 & 1 \\ 2 & 1 & 2 \end{vmatrix} = -5i - 6j + 8k$

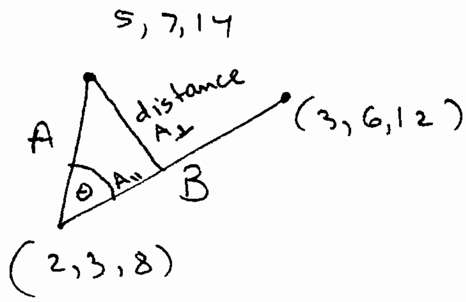
unit vector $\dfrac{-5i-6j+8k}{\pm\sqrt{25+36+64}} = \dfrac{-5i-6j+8k}{\cancel{\pm\sqrt{125}}}$

$\pm 5\sqrt{5}$

$= \pm\frac{1}{25}\sqrt{5}\,(-5i - 6j + 8k)$

1.12 p. 51 cont.

(13) distance from $(5,7,14)$ to line thru $(2,3,8)$, $(3,6,12)$ **Hint:**



$A = 3i + 4j + 6k$

$B = i + 3j + 4k$

$$A \times B = \begin{vmatrix} i & j & k \\ 3 & 4 & 6 \\ 1 & 3 & 4 \end{vmatrix} = -2i - 6j + 5k$$

$$|A_\perp| = \frac{|B \times A|}{|B|} = \frac{\sqrt{4+36+25}}{\sqrt{1+9+16}} = \frac{\sqrt{65}}{\sqrt{26}}$$

(19) $A \cdot B = 0$ & $\Rightarrow$ $|A||B| \cos\theta = 0$

$A \times B = 0$ $\qquad |A||B| \sin\theta = 0$

since one can't have $\sin\theta$ and $\cos\theta$ both zero we must conclude that either A or B is zero.

§1.12

#23

a) $\frac{x}{3} = \frac{y}{2} = \frac{z}{2}$ $\qquad$ $\frac{x}{5} = \frac{y}{3} = \frac{z-4}{2}$

$\vec{R}_1 = t_1\begin{pmatrix}3\\2\\2\end{pmatrix}$ $\qquad$ $\vec{R}_2 = \begin{pmatrix}0\\0\\4\end{pmatrix} + t_2\begin{pmatrix}5\\3\\2\end{pmatrix}$

$\vec{V}_1 = \begin{pmatrix}3\\2\\2\end{pmatrix}$ $\qquad$ $\vec{V}_2 = \begin{pmatrix}5\\3\\2\end{pmatrix}$

note if intersect then

$3t_1 = 5t_2$ and $2t_1 = 3t_2$

$\Downarrow$

$t_1 = \frac{5}{3}t_2 \longrightarrow 2\left(\frac{5}{3}\right)\cancel{t_2} = 3\cancel{t_2}$

$\Rightarrow \frac{10}{3} = 3 \Rightarrow\Leftarrow$

no intersection

b) note line $\perp$ to $\vec{R}_1$ & $\vec{R}_2$ must have

$\vec{V}_3 \perp \vec{V}_1$ & $\vec{V}_2$

let $\vec{V}_3 = \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 2 & 2\\ 5 & 3 & 2\end{vmatrix} = \begin{pmatrix}-2\\4\\-1\end{pmatrix}$

need a line thru $\vec{R}_1$ and $\vec{R}_2$ such that $\vec{R}_1 - \vec{R}_2 = t_3\vec{V}_3$

$$t_1\begin{pmatrix}3\\2\\2\end{pmatrix} - \left[\begin{pmatrix}0\\0\\4\end{pmatrix} + t_2\begin{pmatrix}5\\3\\2\end{pmatrix}\right] = t_3\begin{pmatrix}-2\\4\\-1\end{pmatrix}$$

$3t_1 - 5t_2 = -2t_3$

$2t_1 - 3t_2 = 4t_3$

$2t_1 - 4 - 2t_2 = -t_3 \Rightarrow t_3 = -2t_1 + 2t_2 + 4$

$2t_1 - 3t_2 = 4\left[-2t_1 + 2t_2 + 4\right] = -8t_1 + 8t_2 + 16$

or $\boxed{10t_1 - 11t_2 = 16}$

& $3t_1 - 5t_2 = -2\left[-2t_1 + 2t_2 + 4\right] = 4t_1 - 4t_2 - 8$

$\boxed{-t_1 - t_2 = -8} \rightarrow t_2 = 8 - t_1$

$10t_1 - 11(8 - t_1) = 16$

$21t_1 - 88 = 16 \qquad 21t_1 = 104$

$t_1 = 104/21$

$t_2 = \frac{168}{21} - \frac{104}{21} = \frac{64}{21}$

$t_3 = -\frac{208}{21} + \frac{128}{21} + \frac{84}{21}$

$= 4/21$

only really need t_1 or t_2

$\vec{R}_3 = \frac{104}{21}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$

@ $t=0$ on $\vec{R}_1$ @ $t = t_3$ on $\vec{R}_2$

same as

$$x = \frac{104}{7} - 2t \quad \Rightarrow \quad t = \frac{52}{7} - \frac{x}{2}$$

$$y = \frac{208}{21} + 4t \quad \Rightarrow \quad t = -\frac{52}{21} + \frac{y}{4}$$

$$z = \frac{208}{21} - t \quad \Rightarrow \quad t = \frac{208}{21} - z$$

$$\frac{52}{7} - \frac{x}{2} = -\frac{52}{21} + \frac{y}{4} = \frac{208}{21} - z$$ or negatives of this.

c) To find distances

arb pt on $\vec{R}_1 = \vec{x}_1 = t_1\vec{V}_1$

" " " $\vec{R}_2 = \vec{x}_2 = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + t_2 \vec{V}_2$

distance from $\vec{x}_1$ to $\vec{x}_2$ along direction normal to each is

$$d = \left| (\vec{x}_1 - \vec{x}_2) \cdot \hat{V}_3 \right| \qquad \hat{V}_3 = \frac{1}{\sqrt{21}} \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix}$$

recall $\vec{V}_3 \perp \vec{V}_1$ & $\vec{V}_2$

$$\therefore \quad d = \left| \begin{pmatrix} 0 \\ 0 \\ -4 \end{pmatrix} \cdot \frac{1}{\sqrt{21}} \begin{pmatrix} -2 \\ 4 \\ -1 \end{pmatrix} \right| = 4/\sqrt{21}$$

1.13 p. 57

① a $A = 2i \quad B = 3j \quad C = 5k$

$$[A, B, C] = 2\cdot 3\cdot 5\,[i, j, k] = \underline{\underline{30}}$$

b
$$\begin{vmatrix} 1 & 1 & 1 \\ 3 & 1 & 0 \\ 0 & -1 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} = -3 + 5(1-3) = -3 - 10 = \underline{\underline{-13}}$$

c $[2i - j + k,\ i + j + k,\ 2i + 3k]$

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 3 \end{vmatrix} = 2\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix}$$
$$= 2(-2) + 3(2+1) = -4 + 9 = \underline{\underline{5}}$$

d
$$[k, i, j] = -[i, k, j] = [i, j, k] = \underline{\underline{1}}$$

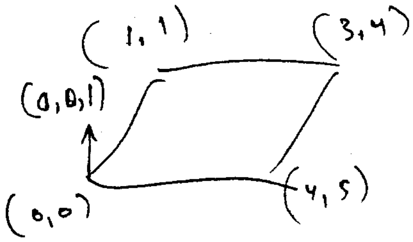
③
$\vec{AB} = i$

$\vec{AC} = -3i - j + 3k$

$\vec{AD} = -3i - 2j + 6k$

$$\begin{vmatrix} 1 & 0 & 0 \\ -3 & -1 & 3 \\ -3 & -2 & 6 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -2 & 6 \end{vmatrix} = -6 + 6 = 0$$

⑤



$\vec{A} = i + j$

$\vec{B} = 4i + 5j$

$\vec{C} = k$

$$\begin{vmatrix} 1 & 1 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 5 - 4 = \underline{\underline{1}}$$

1.13 p.57 cont.

(7) plane parallel to $A = 2i + j + k$ & thru $(3,4,-1)$
$B = i - 3k$

$A \times B$ is normal to plane

$$\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ 1 & 0 & -3 \end{vmatrix} = i(-3) - j(-7) + k(-1)$$

$$= -3i + 7j - k$$

equation of plane $3x - 7y + z = d = 3\cdot 3 - 7\cdot 4 - 1 = 9-28-1$

$\underline{3x - 7y + z = -20}$

1.14 p.60

(5) $\vec{a} = \omega \times (\omega \times R) = (\omega \cdot R)\omega - (\omega \cdot \omega) R$

(6) a $A \times B = B \times A$ no!

b $(A \times B) \times C = A \times (B \times C)$ no!

c $A \times B = A \times C$ iff $B = C$ not necessarily

d $A \times B = 0$ iff $A = 0$ or $B = 0$ — " —

(7) $|A \times B|^2 + (A \cdot B)^2 - |A|^2 |B|^2$

$$\downarrow$$

$$\left(|A|\,|B| \sin\theta\right)^2 + \left(|A|\,|B| \cos\theta\right)^2 - |A|^2 |B|^2$$

$$\downarrow$$

$$\underbrace{|A|^2 |B|^2 \sin^2\theta + |A|^2 |B|^2 \cos^2\theta}_{|A|^2 |B|^2 (\sin^2\theta + \cos^2\theta)} - |A|^2 |B|^2$$

$$|A|^2 |B|^2 (\underbrace{\sin^2\theta + \cos^2\theta}_{=1}) - |A|^2 |B|^2 = 0$$

2.1 p.70

① $F(t) = \sin t\ i + \cos t\ j + k$

ⓐ $F'(t) = \cos t\ i - \sin t\ j$

ⓑ $F'(t)$ parallel to xy plane since no component in z direction.

ⓒ When F' parallel to xz plane

i.e $-\sin t = 0 \Rightarrow t = \pm n\pi \quad n = 0, 1, 2, \ldots$

ⓓ $|F| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$

yes

ⓔ $|F'| = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$

yes

ⓕ $F'' = -\sin t\ i - \cos t\ j$

③ ⓐ $f = (3t\ i + 5t^2 j) \cdot (t i - \sin t\ j)$

$f' = (3i + 10t j) \cdot (t i - \sin t\ j) + (3t i + 5t^2 j) \cdot (i - \cos t\ j)$

$= 3t - 10t \sin t + 3t - 5t^2 \cos t$

$= \underline{6t - 10t \sin t - 5t^2 \cos t}$

ⓑ $f = |\underbrace{2t\ i + 2t\ j - k}_{\vec{F}}| \Rightarrow f^2 = \vec{F} \cdot \vec{F}$

$2 f f' = \frac{d}{dt} \left\{ (2t i + 2t j - k) \cdot (2t i + 2t j - k) \right\}$

$= (2i + 2j) \cdot (2t i + 2t j - k) + (2t i + 2t j - k) \cdot (2i + 2j)$

$= 2 \left[4t + 4t + 0 \right]$

$\Rightarrow f f' = 8t \quad \Rightarrow \quad f' = \dfrac{8t}{|2t i + 2t j - k|} = \dfrac{8t}{\sqrt{4t^2 + 4t^2 + 1}} = \underline{\underline{\dfrac{8t}{\sqrt{1 + 8t^2}}}}$

21 p.70 cont.

(3c) $f = \left[(i+j-2k) \times (\underbrace{3t^4 i + tj}_{\text{only } t \text{ dependent vector}})\right] \cdot k$

$$\frac{df}{dt} = \left[\underbrace{(i+j-2k) \times (12t^3 i + j)}\right] \cdot k$$

$$\begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 12t^3 & 1 & 0 \end{vmatrix} = \underbrace{2i - j(-24t^3) + k(1-12t^3)}_{\text{scalar product of this with } k \text{ yields}}$$

$\underline{1-12t^3}$

(4) Prove $\frac{d}{dt}\left(R \times \frac{dR}{dt}\right) = R \times \frac{d^2R}{dt^2}$

$$\text{LHS} = \underbrace{\frac{d}{dt}R \times \frac{dR}{dt}}_{\frac{dR}{dt} \times \frac{dR}{dt} = 0 \text{ same vector}} + R \times \frac{d^2R}{dt^2}$$

what's left is exactly RHS.

(5) $A = 3i + 2j + 6k$

$B = 3i + 4k$

$C = 2i - 2j + k$

$$\begin{vmatrix} i & j & k \\ 3 & 0 & 4 \\ 2 & -2 & 1 \end{vmatrix} = B \times C \text{ in } 5_i$$

$= 8i + 5j - 6k$

(h) $\frac{d}{dt}(A + Bt) = \underset{=0}{\frac{dA}{dt}} + B \underset{=1}{\frac{dt}{dt}} + \underset{=0}{\frac{dB}{dt}} \cdot t = \underline{B = 3i + 4k}$

const.

(i) $\frac{d}{dt}(B \times tC) = \underset{=0}{\frac{dB}{dt}} \times tC + B \times \left(\underset{=1}{\frac{dt}{dt}} C + t \underset{=0}{\frac{dC}{dt}}\right) = \underline{B \times C = 8i + 5j - 6k}$

2.2 p.85

(1) $x = a\cos t$ $\quad y = b\sin t$ $\quad z = 0$

unit vector . tangent

$$R = a\cos t\, i + b\sin t\, j$$

$$\frac{d\vec{R}}{dt} = -a\sin t\, i + b\cos t\, j$$

$$\left|\frac{d\vec{R}}{dt}\right| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$\left.\frac{\frac{d\vec{R}}{dt}}{\left|\frac{d\vec{R}}{dt}\right|}\right|_{t=3/2\pi} = \frac{-a\overbrace{\sin\tfrac{3}{2}\pi}^{=-1}\,\vec{i} + b\overbrace{\cos\tfrac{3}{2}\pi}^{=0}\,\vec{j}}{\sqrt{a^2\sin^2\tfrac{3}{2}\pi + b^2\cos^2\tfrac{3}{2}\pi}} = \frac{a\vec{i}}{\sqrt{a^2}} = \vec{i}$$

(2) $x = \sin t - t\cos t$ $\quad y = \cos t + t\sin t$ $\quad z = t^2$

(a) arc length $(0,1,0)$ to $(-2\pi, 1, 4\pi^2)$

$$\frac{dx}{dt} = \cancel{\cos t} - [\cancel{\cos t} - t\sin t] = t\sin t$$

$$\frac{dy}{dt} = -\cancel{\sin t} + [\cancel{\sin t} + t\cos t] = t\cos t$$

$$\frac{dz}{dt} = 2t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = t^2\sin^2 t + t^2\cos^2 t + 4t^2$$

$$= t^2(\sin^2 t + \cos^2 t) + 4t^2 = \underline{5t^2}$$

$0 \le t \le 2\pi$ (see z value of endpoints)

$$\int_0^{2\pi} \sqrt{5t^2}\, dt = \sqrt{5}\,\frac{t^2}{2}\Big|_0^{2\pi} = \sqrt{5}\,\frac{4\pi^2}{\cancel{2}}^{2} = 2\sqrt{5}\,\pi^2$$

2.2 P.85 cont.

(2) (b) $T(t) = \frac{dR}{ds}$ or $\frac{\frac{dR}{dt}}{\frac{ds}{dt}} = \frac{\frac{dR}{dt}}{\left|\frac{dR}{dt}\right|}$

$\frac{dR}{dt} = t\sin t\, i + t\cos t\, j + 2t\, k$

from a

$\left|\frac{dR}{dt}\right| = \sqrt{5}\, t$ again from (a)

$\Rightarrow T = \frac{\sin t\, i + \cos t\, j + 2\, k}{\sqrt{5}}$

(c) $T(\pi) = \frac{-j + 2k}{\sqrt{5}}$ since $\sin\pi = 0$

(3) $x = \frac{t}{2\pi}$ $y = \sin t$ $z = \cos t$

$(0,0,1) \to (1,0,1)$ are the endpoints in the example (2.14)

$0 \le t \le 2\pi$

$\frac{d\vec{R}}{dt} = \frac{1}{2\pi}\, i + \cos t\, j - \sin t\, k$

$\left|\frac{d\vec{R}}{dt}\right| = \sqrt{\frac{1}{4\pi^2} + \underbrace{\cos^2 t + (-\sin t)^2}_{=1}} = \sqrt{\frac{1}{4\pi^2} + 1}$

$s = \int_0^{2\pi} \sqrt{\frac{1}{4\pi^2} + 1}\, dt = 2\pi \sqrt{\frac{1}{4\pi^2} + 1} = \cancel{2\pi} \frac{\sqrt{1 + 4\pi^2}}{\cancel{\sqrt{4\pi^2}}}$

$s = \sqrt{1 + 4\pi^2}$

Unit tangent vector at $(0,0,1)$? $\left.\frac{dR}{dt}\right|_{\substack{(0,0,1)\\ t=0}} = \frac{1}{2\pi}\, i + j$ $\Rightarrow T = \frac{i + 2\pi j}{\sqrt{1+4\pi^2}}$

$\left|\frac{dR}{dt}\right|_{t=0} = \sqrt{\frac{1}{4\pi^2} + 1} = \frac{\sqrt{1+4\pi^2}}{2\pi}$

2.2 p.85 cont.

5 (a)

$x = e^t \cos t$ $\qquad \frac{dx}{dt} = e^t \cos t - e^t \sin t$

$y = e^t \sin t$ $\qquad \frac{dy}{dt} = e^t \sin t + e^t \cos t$

$z = 0$ $\qquad \frac{dz}{dt} = 0$

$0 \le t \le 1$

$$s = \int_0^1 \left|\frac{dR}{dt}\right| dt = \int_0^1 \sqrt{e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2 + 0^2}\, dt$$

$$= \int_0^1 e^t \sqrt{\underbrace{\cos^2 t - \cancel{2\sin t \cos t} + \sin^2 t}_{=1} + \sin^2 t + \cancel{2\sin t \cos t} + \cos^2 t}\, dt$$

$$= \sqrt{2}\int_0^1 e^t\, dt = \sqrt{2}\, e^t \Big|_0^1 = \underline{\sqrt{2}(e-1)}$$

(b) To reparametrize we need $s(t)$

$$s(t) = \int_0^t \left|\frac{dR}{dt}\right| dt = \sqrt{2}\int_0^t e^t\, dt = \sqrt{2}(e^t - 1)$$

$$\Rightarrow e^t = \frac{s(t)}{\sqrt{2}} + 1$$

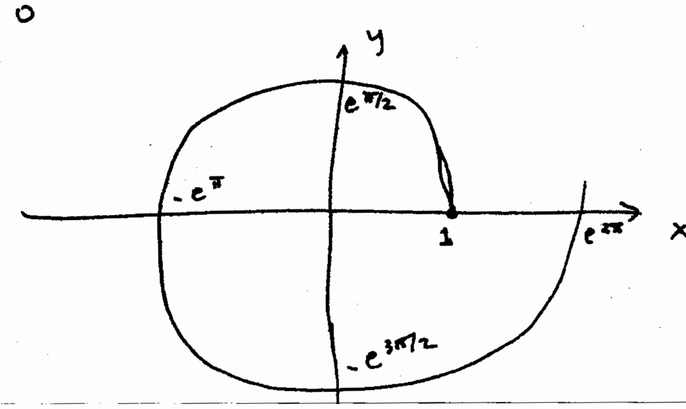
$$t = \ln\left\{\frac{s}{\sqrt{2}} + 1\right\}$$

$$x = \left(\frac{s}{\sqrt{2}} + 1\right)\cos \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$$y = \left(\frac{s}{\sqrt{2}} + 1\right)\sin \ln\left(\frac{s}{\sqrt{2}} + 1\right)$$

$$z = 0$$

(c) sketch



t	x	y
0	1	0
$\pi/2$	0	$e^{\pi/2}$
π	$-e^{\pi}$	0
$\frac{3\pi}{2}$	0	$-e^{3\pi/2}$
2π	$e^{2\pi}$	0

2.2 p.85

(8) $x = t$, $y = 2t^2$, $z = t^3$ intersects the plane $x + 8y + 12z = 162$

at right angle

$$\frac{dR}{dt} = i + 4tj + 3t^2 k$$

for some t $\frac{dR}{dt} \perp$ plane or parallel to normal to plane

i.e. parallel $i + 8j + 12k$

If $t = 2$ then $\left.\frac{dR}{dt}\right|_{t=2} = i + 8j + 12k$

Is that t correspond to intersection?

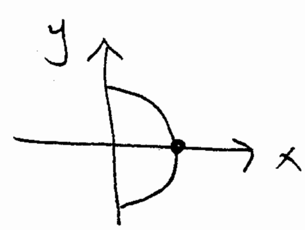
$t = 2 \Rightarrow x = 2, \quad y = 8, \quad z = 8$

$2 + 8 \cdot 8 + 12 \cdot 8 = 162$ and the point is on the plane

$\Rightarrow$ intersection point $(2, 8, 8)$ which is for $t = 2$.

(13) No

For example $x^2 + y^2 = C$ $(x \geq 0)$



$$2x + 2y \frac{dy}{dx} = 0$$

$\frac{dy}{dx} = -\frac{x}{y}$ which does NOT exist for $y = 0$

2.3 p.95

(2) $x = 3t\cos t$ $\quad \dot{x} = 3\cos t - 3t\sin t$ $\quad \ddot{x} = -3\sin t - 3\sin t - 3t\cos t$

$y = 3t\sin t$ $\quad \dot{y} = 3\sin t + 3t\cos t$ $\quad \ddot{y} = 3\cos t + 3\cos t - 3t\sin t$

$z = 4t$ $\quad \dot{z} = 4$ $\quad \ddot{z} = 0$

a $\quad |\vec{v}| = \sqrt{(3\cos t - 3t\sin t)^2 + (3\sin t + 3t\cos t)^2 + 4^2}$

$= \sqrt{\underline{9\cos^2 t} - 18\cancel{t\sin t}\cos t + 9t^2\sin^2 t + \underline{9\sin^2 t} + 18\cancel{t\sin t}\cos t + 9t^2\cos^2 t + 16}$

$= \sqrt{9 + 9t^2 + 16} = \sqrt{9t^2+25} = \frac{ds}{dt}$

b $\quad \vec{a} = (-6\sin t - 3t\cos t)i + (6\cos t - 3t\sin t)j + 0$

$|\vec{a}|^2 = 36\sin^2 t + 36t\sin t\,\cancel{\cos t} + 9t^2\cos^2 t$

$+ 36\cos^2 t - 36t\,\cancel{\sin t}\cos t + 9t^2\sin^2 t$

$= 36 + 9t^2$

$|\vec{a}| = \sqrt{36 + 9t^2}$

$a_T = \frac{d^2 s}{dt^2} = \frac{d}{dt}\sqrt{9t^2+25} = \boxed{\frac{\overset{9}{\cancel{18}}t}{\cancel{2}\sqrt{9t^2+25}} = a_T}$

$a_T^2 + a_N^2 = |\vec{a}|^2 = 36 + 9t^2$

$\downarrow$

$\frac{81t^2}{9t^2+25} + a_N^2 = 36 + 9t^2$

$a_N = \sqrt{36 + 9t^2 - \frac{81t^2}{9t^2+25}}$

c $\quad T = \frac{\vec{v}}{|\vec{v}|} = \frac{(3\cos t - 3t\sin t)i + (3\sin t + 3t\cos t)j + 4k}{\sqrt{9t^2+25}}$

d $\quad \kappa = ?$ $\quad \kappa|\vec{v}|^2 = a_N \Rightarrow \kappa = \frac{\sqrt{36 + 9t^2 - \frac{81t^2}{9t^2+25}}}{9t^2+25} = \frac{3\sqrt{4 + t^2 - \frac{9t^2}{9t^2+25}}}{9t^2+25}$

2.3 p.95 cont.

(5) $R = (\cos t + \sin t)\, i + (\sin t - \cos t)\, j + \frac{1}{2} t k$

a $\dot{R} = (-\sin t + \cos t)\, i + (\cos t + \sin t)\, j + \frac{1}{2} k = \vec{v}$

$|\vec{v}|^2 = (-\sin t + \cos t)^2 + (\cos t + \sin t)^2 + \frac{1}{4}$

$1 - 2\cancel{\sin t}\cos t + 1 + 2\cancel{\sin t}\cos t + \frac{1}{4} = \frac{9}{4}$

$|\vec{v}| = 3/2 = \frac{ds}{dt}$

b $\vec{a} = -(\cos t + \sin t)\, i + (-\sin t + \cos t)\, j$

c $T = \frac{2}{3}(-\sin t + \cos t)\, i + \frac{2}{3}(\cos t + \sin t)\, j + \frac{1}{3} k$

$\kappa = ?$ $\quad a_T = 0$ since $\frac{ds}{dt} = 3/2$

$|\vec{a}|^2 = 1 + 2\cancel{\sin t \cos t} + 1 - \cancel{2\sin t} \cos t = 2$

$|a| = \sqrt{2}$

$\Rightarrow a_N^2 = 2$

$a_N = \sqrt{2} = \kappa \underbrace{|\vec{v}|^2}_{9/4}$

$\Rightarrow \kappa = \frac{\sqrt{2}}{9/4} = \frac{4\sqrt{2}}{9}$ constant.

e Compare to (2.15)

$\underline{e}_1 = i - j$

$\underline{e}_2 = i + j$

$\rho = 1$

$a = 1/2$

(6) $x = 3t^2 - t^3$ $\dot{x} = 6t - 3t^2$ $\ddot{x} = 6 - 6t$

$y = 3t^2$ $\dot{y} = 6t$ $\ddot{y} = 6$

$z = 3t + t^3$ $\dot{z} = 3 + 3t^2$ $\ddot{z} = 6t$

$$\frac{|R' \times R''|}{|R'|^3} = K$$

$$R' \times R'' = \begin{vmatrix} i & j & k \\ 6t-3t^2 & 6t & 3+3t^2 \\ 6-6t & 6 & 6t \end{vmatrix} = i\,(36t^2 - 18 - 18t^2)$$

$$-j\,(36t^2 - \cancel{18t^3} - 18 + 18t - 18t^2 + \cancel{18t^3})$$

$$+k\,(\cancel{36t} - 18t^2 - \cancel{36t} + 36t^2)$$

$$= 18(t^2-1)\,i - 18(t^2+t-1)\,j + 18t^2 k$$

$$|R' \times R''| = \sqrt{18^2(t^2-1)^2 + 18^2(t^2+t-1)^2 + 18^2 t^4}$$

$$= 18\sqrt{t^4 - 2t^2 + 1 + t^4 + 2t^3 - 2t^2 + t^2 - 2t + 1 + t^4}$$

$$= 18\sqrt{3t^4 + 2t^3 - 3t^2 - 2t + 2}$$

$$|R'|^2 = 36t^2 - 36t^3 + 9t^4 + 36t^2 + 9 + 18t^2 + 9t^4$$

$$= 18t^4 - 36t^3 + 90t^2 + 9$$

$$= 9(2t^4 - 4t^3 + 10t^2 + 1)$$

$$K = \frac{\overset{2}{\cancel{18}}\sqrt{3t^4 + 2t^3 - 3t^2 - 2t + 2}}{\underset{3}{\cancel{3^3}}\,(2t^4 - 4t^3 + 10t^2 + 1)^{3/2}}$$

2.3 p.95 cont.

7) $R(t) = \underbrace{\sin t}_{x}\, i + \underbrace{\cos t}_{y}\, j + \underbrace{\log \sec t}_{z}\, k \qquad 0 \le t \le \pi/2$

a find ds

$$ds^2 = dx^2 + dy^2 + dz^2 = (\cos t\, dt)^2 + (-\sin t\, dt)^2 + \left(\frac{1}{\cancel{\sec t}} \tan t \cancel{\sec t}\, dt\right)^2$$

$$ds = \sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1} + \tan^2 t}\; dt = \sec t\, dt$$

but $1 + \tan^2 t = \sec^2 t$

b $T = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\cos t\, i - \sin t\, j + \tan t\, k}{\sec t}$

c $N = \dfrac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|}$

$$\frac{dT}{dt} = \left(\frac{\cos t}{\sec t}\right)' i - \left(\frac{\sin t}{\sec t}\right)' j + \underbrace{\left(\frac{\tan t}{\sec t}\right)'}_{=(\sin t)'} k$$

$$= -2\cos t \sin t\, i - \underbrace{(\cos^2 t - \sin^2 t)}_{\cos 2t}\, j + \cos t\, k$$

$$= -\sin 2t\, i - \cos 2t\, j + \cos t\, k$$

$$N = \frac{-\sin 2t\, i - \cos 2t\, j + \cos t\, k}{\sqrt{\underbrace{\sin^2 2t + \cos^2 2t}_{=1} + \cos^2 t}} =$$

$$N = -\frac{\sin 2t\, i + \cos 2t\, j - \cos t\, k}{\sqrt{1 + \cos^2 t}}$$

d. $K = ?$

Use: $K = \dfrac{|R' \times R''|}{|R'|^3}$ or find components of $\vec{a}$

2.3 p. 95 cont.

7d cont.

$R = \sin t \ i + \cos t \ j + \log \sec t \ k$

$R' = \vec{v} = \cos t \ i - \sin t \ j - \tan t \ k$ $\qquad |\vec{v}| = \frac{ds}{dt} \underset{\substack{\uparrow \\ \text{part a}}}{=} \sec t$

$R'' = \vec{a} = -\sin t \ i - \cos t \ j - \sec^2 t \ k$

$a_T = \frac{d^2 s}{dt^2} \underset{\substack{\uparrow \\ \text{differentiate } (\sec t)}}{=} \tan t \sec t$

$a_N^2 = |\vec{a}|^2 - a_T^2 = (-\sin t)^2 + (-\cos t)^2 + (-\sec^2 t)^2 - \tan^2 t \sec^2 t$

$= \sin^2 t + \cos^2 t + \sec^4 t - \sec^2 t \tan^2 t$

$= 1 + \sec^2 t \underbrace{(\sec^2 t - \tan^2 t)}_{=1}$

$= 1 + \sec^2 t$

$K |\vec{v}|^2 = a_N = \sqrt{1 + \sec^2 t}$

$$K = \frac{\sqrt{1 + \sec^2 t}}{\sec^2 t}$$

2.3 p. 95 cont

(9) $R = \log(t^2+1)\, i + (t - 2\arctan t)\, j + 2\sqrt{2}\, t\, k$

$$\vec{v} = \frac{1}{t^2+1}\cdot 2t\, i + \Big(\underbrace{1 - \frac{2}{1+t^2}}_{\frac{\cancel{1}+t^2-\cancel{2}^{\,1}}{1+t^2}}\Big) j + 2\sqrt{2}\, k$$

$$|\vec{v}| = \frac{4t^2}{(t^2+1)^2} + \frac{(t^2-1)^2}{(1+t^2)^2} + (2\sqrt{2})^2$$

$$= \frac{\overset{2t^2}{\cancel{4t^2}} + t^4 \cancel{-2t^2} + 1}{(t^2+1)^2} + 8$$

$$= \frac{(t^2+1)^2}{(t^2+1)^2} + 8 = 1 + 8 = 9 \quad \underline{\text{constant}}$$

$$\frac{ds}{dt} = \text{constant} \Rightarrow \frac{d^2s}{dt^2} = 0 \Rightarrow a_T = 0$$

b

$$K = ? = \frac{a_N}{|\vec{v}|^2} = \frac{|\vec{a}|}{|\vec{v}|^2}$$

$$\vec{a} = \left(\frac{2t}{t^2+1}\right)' i + \left(\frac{t^2-1}{t^2+1}\right)' j$$

$$= \frac{2(t^2+1) - 2t\cdot 2t}{(t^2+1)^2}\, i + \frac{2t(\cancel{t^2}+1) - (\cancel{t^2}-1)2t}{(t^2+1)^2}\, j$$

$$= \frac{(\cancel{2t^2}+1 - \overset{2}{\cancel{4}}t^2)\, i + 4t\, j}{(t^2+1)^2}$$

$$|\vec{a}|^2 = \frac{(1-2t^2)^2 + (4t)^2}{(t^2+1)^4} = \frac{1 \cancel{-4t^2} + 4t^4 + \overset{12}{\cancel{16}}t^2}{(t^2+1)^4}$$

$$K = \frac{\sqrt{1+12t^2+4t^4}}{(t^2+1)^2}\cdot\frac{1}{9^2} = \underline{\underline{\frac{\sqrt{1+12t^2+4t^4}}{81(t^2+1)^2}}}$$

2.3 p.95 cont.

(15) a $\underbrace{\frac{dR}{ds}}_{T} \cdot T = |T|^2 = 1$ unit vector

b $\frac{d}{ds}(T \cdot T) = 0$ since $\frac{dT}{ds}$ parallel to $N \perp T$

c $\frac{d^2R}{dt^2} \cdot T = a \cdot T = a_T$

d $T \cdot N = 0$

e $\frac{dR}{dt} \cdot T = \frac{ds}{dt}\frac{dR}{ds} \cdot T = \underbrace{\frac{ds}{dt}}_{|\vec{v}|} \underbrace{|T|^2}_{=1} = |\vec{v}|$.

f $\frac{dN}{ds} \cdot B = \tau$

g $[T, N, B] = 1$ right hand system

h $\left|\frac{d^2R}{ds^2}\right| = \kappa$ since $\left|\frac{d^2R}{ds^2}\right| = \left|\frac{d}{ds}\underbrace{\frac{dR}{ds}}_{T}\right| = \left|\frac{dT}{ds}\right| = \kappa|N| = \kappa$

i $\frac{dB}{ds} = -\tau N$ (Frenet formula)

(17) $x = \cos^3 t \qquad y = \sin^3 t \qquad z = 2\sin^2 t \qquad 0 < t \le \pi/2$

$$T = \frac{dR}{ds} = \frac{dR/dt}{ds/dt} = \frac{-3\cos^2 t \sin t \, i + 3\sin^2 t \cos t \, j + 4\sin t \cos t \, k}{\sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2 + 16\sin^2 t \cos^2 t}}$$

$$\text{den} = \sqrt{9\cos^4 t \sin^2 t + 9\sin^4 t \cos^2 t + 16\sin^2 t \cos^2 t}$$
$$= \sqrt{\cos^2 t \sin^2 t\,(\underbrace{9\cos^2 t + 9\sin^2 t}_{9} + 16)} = 5\sin t \cos t$$

$$\boxed{T = \frac{-3\cos t \, i}{5} + \frac{3}{5}\sin t \, j + \frac{4}{5} k}$$

17 cont.

$$N = \frac{\frac{dT}{dt}}{\left|\frac{dT}{dt}\right|} = \frac{+\frac{3}{5}\sin t \; i + \frac{3}{5}\cos t \; j}{\sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}} = \frac{5}{3}\left[\frac{3}{5}\sin t \; i + \frac{3}{5}\cos t \; j\right]$$

$$\underline{N = \sin t \; i + \cos t \; j}$$

$$B = T \times N = \begin{vmatrix} i & j & k \\ -3/5 \cos t & 3/5 \sin t & 4/5 \\ \sin t & \cos t & 0 \end{vmatrix} = i\left(-\tfrac{4}{5}\cos t\right) - j\left(-\tfrac{4}{5}\sin t\right) + k\underbrace{\left(-\tfrac{3}{5}\cos^2 t - \tfrac{3}{5}\sin^2 t\right)}_{-3/5}$$

$$\underline{= -\frac{4}{5}\cos t \; i + \frac{4}{5}\sin t \; j - \frac{3}{5} k}$$

$$\kappa = \left|\frac{dT}{ds}\right| = \left|\frac{dT}{dt}\right|\frac{dt}{ds} = \frac{\left|\frac{dT}{dt}\right|}{\frac{ds}{dt}} = \frac{\left|\frac{dT}{dt}\right|}{\frac{ds}{dt}} = \frac{3/5 \text{ (see comp. N)}}{5 \sin t \cos t \text{ (see comp. T)}}$$

$$\underline{\kappa = \frac{3}{25 \sin t \cos t}}$$

$$\tau = \pm\left|\frac{dB}{ds}\right| = \pm\frac{\left|\frac{dB}{dt}\right|}{\frac{ds}{dt}} = \pm\frac{\sqrt{\left(+\frac{4}{5}\sin t\right)^2 + \left(\frac{4}{5}\cos t\right)^2 + 0^2}}{5 \sin t \cos t}$$

$$= \pm\frac{\frac{4}{5}}{5 \sin t \cos t} = \boxed{\pm\frac{4}{25 \sin t \cos t} = \tau}$$

Compare with (2.45) to find sign to be <u>negative</u>

2.4 p.102

(1) $r = b(1-\cos\theta)$ $\qquad \dfrac{dr}{d\theta} = +b\sin\theta$

$\dfrac{d\theta}{dt} = 4 \Longrightarrow \dfrac{d^2\theta}{dt^2} = 0$

$$\vec{v} = \underbrace{\frac{dr}{dt}}_{\underbrace{\frac{dr}{d\theta}}_{b\sin\theta}\underbrace{\frac{d\theta}{dt}}_{4}} u_r + r\underbrace{\frac{d\theta}{dt}}_{4} u_\theta = 4b\sin\theta\, u_r + 4b(1-\cos\theta)\,u_\theta$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \left[\underbrace{\frac{d^2r}{dt^2}}_{} - r\underbrace{\left(\frac{d\theta}{dt}\right)^2}_{16}\right]u_r + \left[r\underbrace{\frac{d^2\theta}{dt^2}}_{=0} + 2\underbrace{\frac{dr}{dt}}_{4b\sin\theta}\underbrace{\frac{d\theta}{dt}}_{=4}\right]u_\theta$$

$$= \frac{d}{dt}(4b\sin\theta)$$

$$= 4b\cos\theta\underbrace{\frac{d\theta}{dt}}_{=4}$$

$$\vec{a} = \left[16b\cos\theta - 16b(1-\cos\theta)\right]u_r + 32b\sin\theta\, u_\theta$$

$$\vec{a} = 16b(2\cos\theta - 1)\,u_r + 32b\sin\theta\, u_\theta$$

(3) $r = 2(1+\sin\theta) \Rightarrow \dfrac{dr}{d\theta} = 2\cos\theta \Rightarrow \dfrac{dr}{dt} = 2\cos\theta\dfrac{d\theta}{dt} = -2\cos\theta\, e^{-t}$

$\theta = e^{-t} \Rightarrow \dfrac{d\theta}{dt} = -e^{-t}$

$$\vec{v} = \frac{dr}{dt}u_r + r\frac{d\theta}{dt}u_\theta = -2\cos\theta\, e^{-t}u_r - 2(1+\sin\theta)e^{-t}u_\theta$$

(4)
$$r = 2+\sin t \qquad \frac{dr}{dt} = \cos t$$

$$\vec{v} = \frac{dr}{dt}\,u_r + r\frac{d\theta}{dt}\,u_\theta = \underline{\cos t}\;u_r + \underline{(2+\sin t)\frac{d\theta}{dt}}\;u_\theta$$

$$|\vec{v}|^2 = \cos^2 t + (2+\sin t)^2\left(\frac{d\theta}{dt}\right)^2 = 2\cos^2 t = v^2 \text{ given}$$

$$(2+\sin t)^2\left(\frac{d\theta}{dt}\right)^2 = \cos^2 t$$

$$\frac{d\theta}{dt} = \frac{\cos t}{2+\sin t}$$

$$\int d\theta = \int \frac{\cos t}{2+\sin t}\,dt = \int \frac{dx}{x} = \ln x + \text{const.} = \ln|2+\sin t| + \text{const}$$

Let $x = 2+\sin t$
$dx = \cos t\,dt$

$$\theta = \ln|2+\sin t| + \text{const.}$$

but $\theta = 0$ at $t=0$ $\Rightarrow$ $0 = \ln|2+\sin 0| + \text{const.}$

$$c = -\ln 2$$

$$\Rightarrow \underline{\theta} = \ln\frac{2+\sin t}{2} = \underline{\ln\left(1+\tfrac{1}{2}\sin t\right)}$$

at $t = \pi/2$ $\qquad \theta(\pi/2) = \ln\left(1+\tfrac{1}{2}\underbrace{\sin\pi/2}_{=1}\right) = \ln 3/2$

$$r(\pi/2) = 2+\sin\pi/2 = 3$$

position $\underline{(3, \ln 3/2) = (r,\theta)}$

2.4 p. 102 cont.

(6)

$$a = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]u_r + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]u_\theta$$

a moving around circle at (0,0) with constant nonzero angular velocity $\frac{d\theta}{dt} = c$ $\quad \frac{d^2\theta}{dt^2} = 0 \quad \frac{dr}{dt} = 0$ since $r = \text{const.}$

$\Rightarrow$

$$a = -r\left(\frac{d\theta}{dt}\right)^2 u_r$$

b moving around circle at (0, 0) with constant nonzero angular acceleration $\frac{d^2\theta}{dt^2} = c$, $\frac{dr}{dt} = 0$

$$a = -r\left(\frac{d\theta}{dt}\right)^2 u_r + r\frac{d^2\theta}{dt^2}u_\theta$$

c moves along a straight line with constant speed

$\theta = \text{const.}$ $\Rightarrow$ all terms are zero

d Depends

(10) constant radial speed $= \frac{dr}{dt} = 2\,\frac{\text{cm}}{\text{sec}}$

platform rotating with uniform angular velocity $30\,\frac{\text{rev}}{\text{min}}$

$$30\,\frac{\text{rev}}{\text{min}} = \cancel{30}\,\frac{\text{rev}}{\underset{2}{\cancel{60}}\,\text{sec}} = \frac{1}{2}\,\frac{\text{rev}}{\text{sec}} = \pi\,\frac{\text{rad}}{\text{sec}}$$

a. radial acceleration

$$a_r = \underbrace{\frac{d^2r}{dt^2}}_{=0} - r\Big(\underbrace{\frac{d\theta}{dt}}_{\pi}\Big)^2 = -\pi^2 r$$

means toward center

b. coriolis acceleration $= 2\underbrace{\frac{dr}{dt}}_{2}\underbrace{\frac{d\theta}{dt}}_{\pi} = 4\pi\,\frac{\text{cm}}{\text{sec}}$

3.1 p.112

(1) a $f = \sin x + e^{xy} + z$

$$\frac{\partial f}{\partial x} = \cos x + y e^{xy} + 0$$

$$\frac{\partial f}{\partial y} = 0 + x e^{yx} + 0$$

$$\frac{\partial f}{\partial y} = 0 + 0 + 1$$

$$\text{grad } f = (\cos x + y e^{xy})\, i + x e^{xy}\, j + k$$

b $f = \dfrac{1}{|R|} = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$

$$\frac{\partial f}{\partial x} = \frac{-\frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2} \cdot \cancel{2}x}{x^2+y^2+z^2} = -\frac{x}{(x^2+y^2+z^2)^{3/2}}$$

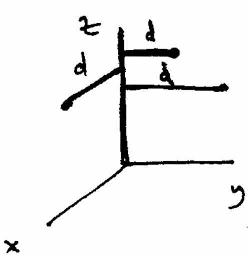
$$\frac{\partial f}{\partial y} = -\frac{y}{(\quad)^{3/2}} \quad ; \quad \frac{\partial f}{\partial z} = -\frac{z}{(\quad)^{3/2}}$$ using symmetry

$$\text{grad } f = -\frac{R}{|R|^3}$$

c $f = R \cdot i \times j = (x i + y j + z k) \cdot \underbrace{i \times j}_{k} = z$

$$\text{grad } f = k$$

(6)



Unit vector directed away from the z axis except at points on the z axis where it is not defined

$$\left(f = \sqrt{x^2 + y^2 + 0} \right)$$

3.1 p.112 cont.

7 $f = x^2 + y^2 + z^2$

a grad $f = 2x\,i + 2y\,j + 2z\,k$ at $(3,0,4)$ grad $f = 6i + 8k$

$|\text{grad } f| = \sqrt{36+64} = \underline{\underline{10}}$

$\frac{df}{ds} = \text{grad } f \cdot \frac{d\bar{x}}{ds} = |\text{grad } f|\,|1|\cos\theta$

3.1 p. 112 cont.

(8) $z = h e^{-(x^2+2y^2)}$

$f = -h e^{-(x^2+2y^2)} + z$

a. $\nabla f = 2x h e^{-(x^2+2y^2)} i + 4y h e^{-(x^2+2y^2)} j + k$

at $(1, 2, h e^{-9})$

lava flow steepest descent $\Rightarrow$ grad f

$$\nabla f \Big|_{(1,2,he^{-9})} = 2 h e^{-9} i + 8 h e^{-9} j + k$$

direction: $\dfrac{\nabla f}{|\nabla f|} = \dfrac{2he^{-9} i + 8he^{-9} j + k}{\sqrt{4h^2 e^{-18} + 64 h^2 e^{-18} + 1}}$

Projection on the xy plane: $2he^{-9} i + 8he^{-9} j$

b. curve $\dfrac{x-1}{2he^{-9}} = \dfrac{y-2}{8he^{-9}} = \dfrac{z-he^{-9}}{1}$ in the direction of ∇f.

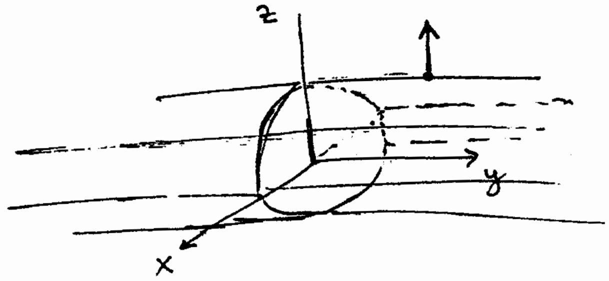
The projection on the xy plane:

$$\frac{x-1}{2he^{-9}} = \frac{y-2}{8he^{-9}} \Rightarrow 4(x-1) = y-2$$

$$y = 4x - 2$$

(16) $x^2+z^2=8$ at $(2,0,2)$ cylinder

always normal to y axis!



b $f = x^2+z^2$

$\text{grad } f\Big|_{2,0,2} = 2x\,i + 2z\,k\Big|_{2,0,2} = \underline{\underline{4i + 4k}}$ normal vector

no component in y direction

$\Rightarrow \perp$ y axis

3.1 p. 112 contine

(17) plane tangent to $f = z^2 - xy - 14$ at 2,1,4

grad $f\Big|_{2,1,4} = -y\,i - x\,j + 2z\,k\Big|_{2,1,4} = -i - 2j + 8k$

normal to surface => normal to tangent plane

eq. of plane $-x - 2y + 8z = C$

at 2,1,4 $-2 - 2 + 32 = C \Rightarrow C = 28$

$$\underline{-x - 2y + 8z = 28}$$

(21) $T = x^2 + 2y^2 + 3z^2$

S isotimic surface $T = 1$

$S: x^2 + 2y^2 + 3z^2 = 1$

$2x\,i + 4y\,j + 6z\,k$ vector normal to S => normal to tangent plane

at (x, y, z) we want the normal (1,1,1)

$\Rightarrow 4y = 2x \quad 6z = 2x$

$$\boxed{x = 2y = 3z}$$

Plug in T = 1

$$x^2 + 2\left(\frac{x}{2}\right)^2 + 3\left(\frac{x}{3}\right)^2 = 1$$

$$\Rightarrow x^2 = \frac{6}{11} \Rightarrow x = \pm\sqrt{\frac{6}{11}}$$

$$\Rightarrow y = \frac{1}{2}x = \pm\frac{1}{2}\sqrt{\frac{6}{11}} \quad ; \quad z = \pm\frac{1}{3}\sqrt{\frac{6}{11}}$$

3.1 p.112 cont.

24 $\nabla f = 2x\,i + 2y\,j + 2z\,k \Big|_{(\frac{1}{2}, \frac{3}{2}, 3\sqrt{6}/2)} = i + 3j + 3\sqrt{6}\,k$

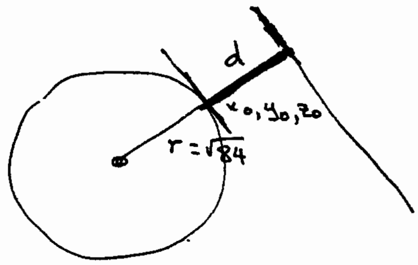
$\nabla g = 2(x-1)\,i + 2y\,j + 2z\,k \Big|_{(\frac{1}{2}, \frac{3}{2}, \frac{3\sqrt{6}}{2})} = -i + 3j + 3\sqrt{6}\,k$

$$\cos\theta = \frac{\nabla f \cdot \nabla g}{|\nabla f|\,|\nabla g|} = \frac{-1 + 9 + 9\cdot 6}{\sqrt{1+9+9\cdot 6}\,\sqrt{1+9+9\cdot 6}} = \frac{62}{\sqrt{64}\,\sqrt{64}} = \frac{31}{32}$$

$\theta = \arccos \frac{31}{32}$

3.1 p.112 cont.

(31) $x^2+y^2+z^2=84$ closest $x+2y+4z=77$



Find tangent plane parallel given plane:

grad $f = 2xi+2yj+2zk$

Find (x_0,y_0,z_0) such that this normal is parallel to $i+2j+4k$

$\Rightarrow x_0=\frac{1}{2}\alpha \quad y_0=\alpha \quad z_0=2\alpha$

Since the point is on sphere

we have: $\left(\frac{1}{2}\alpha\right)^2+\alpha^2+(2\alpha)^2=84$

$$\underbrace{\left(\frac{1}{4}+1+4\right)}_{21/4}\alpha^2=84$$

$$\alpha^2=\frac{\cancel{84}^{\,4}}{\cancel{21}}\cdot 4=16$$

$$\underline{\underline{\alpha=4}}$$

$\Rightarrow (x_0,y_0,z_0)=(2,4,8)$

If you need: (not asked for)
distance from point to plane $=\dfrac{|2+2\cdot4+4\cdot8-77|}{\sqrt{1+4+16}}$

$$=\frac{|2+8+32-77|}{\sqrt{21}}=\frac{15}{\sqrt{21}}$$

3.2 p. 117

① $F = -y\,i + x\,j$

$F(1,0) = j$

$F(0,1) = -i$

$F(-1,0) = -j$

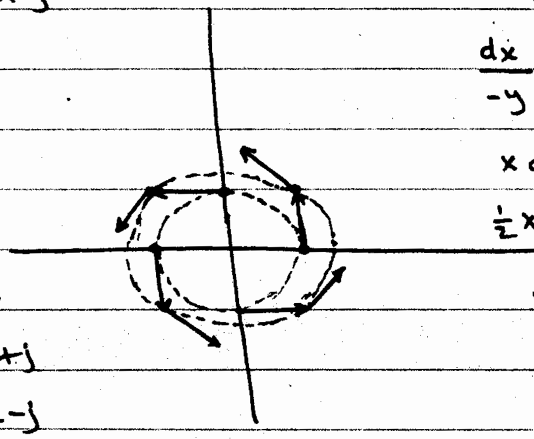
$F(0,-1) = i$

$F(1,1) = -i+j$

$F(-1,1) = -i-j$

$F(-1,-1) = i-j$

$F(1,-1) = i-j$



$\frac{dx}{-y} = \frac{dy}{x}$

$x\,dx = -y\,dy$

$\frac{1}{2}x^2 = -\frac{1}{2}y^2 + \text{const.}$

$x^2 + y^2 = \text{const}$

circles

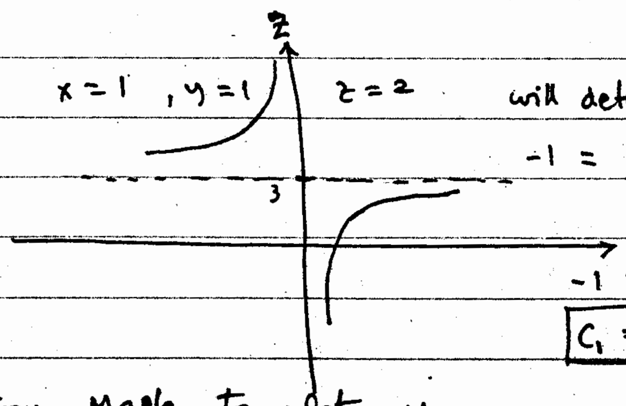
② $F = x^2 i + y^2 j + k$

ⓐ $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{1}$

$-\frac{1}{y} = z + \text{cons.} \Rightarrow \boxed{-1 = y(z + C_2)}$

$\boxed{-1 = x(z - C_1)} \Leftarrow -\frac{1}{x} = z - C_1$

ⓑ $x = 1$, $y = 1$, $z = 2$ will determine C_1, C_2



$-1 = 1(2 - C_1)$

$\Downarrow$

$-1 = 2 - C_1$

$\boxed{C_1 = 3}$

$-1 = 1(2 + C_2)$

$-1 = 2 + C_2$

$\boxed{-3 = C_2}$

Try Maple to plot these $\Rightarrow$ $\left\{ \begin{array}{l} x(z-3) = -1 \\ y(z-3) = -1 \end{array} \right\}$

3.2 p.117 cont.

3. $R = xi + yj + zk$

describe flow lines -

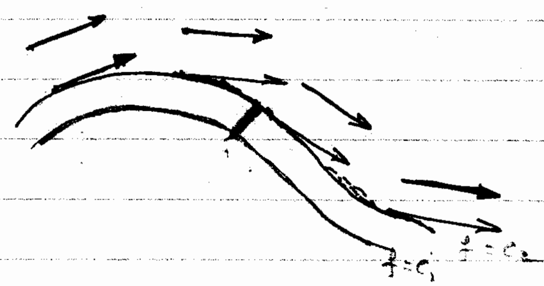
$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$\ln x = \ln y + \ln c$ similarly $y = cz$

$x = cy$

which are lines away from origin.

4. The flow lines of the gradient field cross isotimic surfaces orthogonally:



The distance between isotimic surfaces is constant and it is the normal to surfaces which is perpendicular to gradient

3.3 p. 124

(1) $F = e^{xy}\, i + \sin(xy)\, j + \cos^2(zx)\, k$

$\operatorname{div} F = ye^{xy} + x\cos xy + x \cdot 2\cos zx\,(-\sin zx)$

(3) $F = \operatorname{grad} 3x^2y^3z$

$F = 6xy^3z\, i + 9x^2y^2z\, j + 3x^2y^3\, k$

$\operatorname{div} F = 6y^3z + 18x^2yz + 0 = \underline{6yz(y^2+x^2)}$

(5) $\operatorname{div}(\varphi F) = \varphi \operatorname{div} F + F \cdot \operatorname{grad} \varphi$

$\downarrow$

$$\frac{\partial}{\partial x}(\varphi F_1) + \frac{\partial}{\partial y}(\varphi F_2) + \frac{\partial}{\partial z}(\varphi F_3) =$$

$$\frac{\partial \varphi}{\partial x} F_1 + \varphi \frac{\partial F_1}{\partial x} + \frac{\partial \varphi}{\partial y} F_2 + \varphi \frac{\partial F_2}{\partial y} + \frac{\partial \varphi}{\partial z} F_3 + \varphi \frac{\partial F_3}{\partial z}$$

$$\underbrace{\frac{\partial \varphi}{\partial x} F_1 + \frac{\partial \varphi}{\partial y} F_2 + \frac{\partial \varphi}{\partial z} F_3}_{\operatorname{grad}\varphi \cdot \vec{F}} + \varphi\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)$$

$\operatorname{grad}\varphi \cdot \vec{F} \quad + \quad \varphi \cdot \operatorname{div} F \qquad \square$

(7) Nonconstant field with zero divergence

$$F = f(y,z)\, i + g(x,z)\, j + h(x,y)\, k$$

$$\Rightarrow \quad \frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y} = \frac{\partial F_3}{\partial z} = 0$$

(10) The divergence is zero (length of arrows identical)

3.4 p.132

(1) $F = xy^2 i + xy j + xy k$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & xy & xy \end{vmatrix} = i(x-0) - j(y-0) + k(y - 2xy)$$

$$= \underline{xi - yj + y(1-2x)k}$$

(2) $F = e^{xy} i + \sin xy\, j + \cos yz^2 k$

$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{xy} & \sin xy & \cos yz^2 \end{vmatrix} = i(-z^2 \sin yz^2 - 0) - j(0-0) + k(y\cos xy - x e^{xy})$$

$$= \underline{-z^2 \sin yz^2\; i + (y \cos xy - x e^{xy}) k}$$

(4) $F = (x + xz^2) i + xy j + yz k$

a div $F = (1 + z^2) + y + y = \underline{1 + 2y + z^2}$

b
$$\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + xz^2 & xy & yz \end{vmatrix} = i \underbrace{\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz \end{vmatrix}}_{z - 0} - j(0 - 2xz) + k(y - 0)$$

$$= \underline{z i + 2xz j + y k}$$

(7) The flow lines of a velocity field F are straight lines, this does <u>NOT</u> mean the curl is zero

3.4 P.132 cont.

(10) $F = y^2 i + z^2 j + xk$

a $\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x \end{vmatrix} = i(0-2z) - j(1-0) + k(0-2y)$

$$= \underline{-2z\,i - j - 2y\,k}$$

b $x = \cos\pi t \quad y = \sin\pi t \quad z = t^2$

Tangent to this curve when $t=1$

$\frac{\partial x}{\partial t} = -\pi\sin\pi t \quad \frac{\partial y}{\partial t} = \pi\cos\pi t \quad \frac{\partial z}{\partial t} = 2t$

$$T = \frac{-\pi\sin\pi t\; i + \pi\cos\pi t\; j + 2t\,k}{\sqrt{(\pi\sin\pi t)^2 + (\pi\cos\pi t)^2 + 4t^2}}$$

$$T = \frac{-\pi\sin\pi t\; i + \pi\cos\pi t\; j + 2t\,k}{\sqrt{\pi^2 + 4t^2}}$$

at $t=1 \Rightarrow x=-1, \; y=0, \; z=1$

$$T(t=1) = \frac{-\pi j + 2k}{\sqrt{\pi^2+4}}$$

$$\underline{\text{curl } F \cdot T(t=1)} = |\text{curl } F|\, \underbrace{|T(t=1)|}_{=1} \cos\theta$$

$$\frac{\text{curl } F \cdot T(t=1)}{\cancel{|T(t=1)|}} = \frac{(-2i - j - 0k)\cdot(-\pi j + 2k)}{\sqrt{\pi^2+4}} = \underline{\underline{\frac{\pi}{\sqrt{\pi^2+4}}}}$$

$$(\nabla\times\bar{F})\cdot \frac{d\bar{r}}{dt} = |\nabla\times\bar{F}|\left|\frac{d\bar{r}}{dt}\right|\cos\theta$$

$$\frac{\bar{\nabla}\times\bar{F}\cdot\frac{d\bar{r}}{dt}}{\left|\frac{d\bar{r}}{dt}\right|} = |\bar{\nabla}\times\bar{F}|\cos\theta$$

3.4 p. 132 cont.

(12) curl $\left[f(|R|)\vec{R} \right]$ $\quad \vec{R} = xi + yj + zk$

a

$$\operatorname{curl}\left(f(|R|)\vec{R}\right) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(|R|)x & f(|R|)y & f(|R|)z \end{vmatrix}$$

$$= i\left(z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z}\right) - j\left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)$$

$$+ k\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial |R|}\frac{\partial |R|}{\partial x} = \frac{\partial f}{\partial |R|}\cancel{\tfrac{1}{2}}(x^2+y^2+z^2)^{-1/2}\cdot\cancel{2}x$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial |R|}\frac{\partial |R|}{\partial y} = \frac{\partial f}{\partial |R|}\, y(x^2+y^2+z^2)^{-1/2}$$

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial |R|}\frac{\partial |R|}{\partial z} = \frac{\partial f}{\partial |R|}\, z(x^2+y^2+z^2)^{-1/2}$$

$$\Rightarrow z\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial z} = zy(\quad)^{-1/2}\frac{\partial f}{\partial |R|} - yz(\quad)^{-1/2}\frac{\partial f}{\partial |R|} = 0$$

similarly for other terms

$$\Rightarrow \underline{\operatorname{curl}\left(f(|\vec{R}|)\vec{R}\right) = 0\,!}$$

3.4 p.132 cont.

12b. Use geometrical interpretation

3:5 p. 135

(1) $f = x^2 y + z$

$f(2,3,4) = 2^2 \cdot 3 + 4 = \underline{16}$

(2) $f = x^2 y + z$

$\nabla f = 2xy\, i + x^2 j + k$

$\nabla f\big|_{2,3,4} = 2 \cdot 2 \cdot 3\, i + 2^2 j + k = \underline{12i + 4j + k}$

(4) $F = x^2 y\, i + z\, j - (x+y-z) k$

a $\nabla \cdot F = 2xy + 0 + 1 = \underline{2xy + 1}$

b $$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & z & -(x+y-z) \end{vmatrix} =$$

$$i \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -(x+y-z) \end{vmatrix} - j \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2 y & -(x+y-z) \end{vmatrix} + k \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 y & z \end{vmatrix}$$

$= i(-1-1) - j(-1+0) + k(0 - x^2)$

$= -2i + \underline{j - x^2 k}$

c $\nabla(\underbrace{\nabla \cdot F}_{\text{see } a}) = \text{grad}\,(2xy + 1) = \underline{2y\, i + 2x\, j}$

(5) $\underbrace{\nabla \cdot \underbrace{(\nabla \times F)}_{\text{vector}}}_{\underline{\text{scalar}}}$

(6) $\underbrace{\nabla \times \underbrace{(\nabla \times F)}_{\text{vector}}}_{\text{vector}}$

3.5 p.135 cont.

(7) $R = x\,i + y\,j + z\,k$

$\nabla \cdot R = 1 + 1 + 1 = \underline{\underline{3}}$

$$\nabla \times R = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = i \underbrace{\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix}}_{=0} - j \underbrace{\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix}}_{=0}$$

$$+ k \underbrace{\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}}_{=0} = \underline{\underline{\vec{0}}}$$

(8) $f = xyz + e^{xz}$

$\nabla \cdot \nabla f = ?$

$\nabla f = \left(yz + z e^{xz}\right) i + xz\,j + \left(xy + x e^{xz}\right) k$

$\nabla \cdot \nabla f = z^2 e^{xz} + 0 + x^2 e^{xz} = \underline{(x^2+z^2) e^{xz}}$

3.6 p. 140

(1) $f = x^5 y z^3$

$$\frac{\partial^2 f}{\partial x^2} = 5\cdot 4\cdot x^3\, y\cdot z^3$$

$$\frac{\partial^2 f}{\partial y^2} = 0$$

$$\frac{\partial^2 f}{\partial z^2} = 3\cdot 2\, x^5 y z$$

$$\nabla^2 f = \underline{20\, x^3 y z^3 + 6 x^5 y z = 2 x^3 y z\,(10 z^2 + 3x^2)}$$

(3) $\vec{F} = 3i + j - x^2 y^3 z^4 k$

$$\frac{\partial^2 \vec{F}}{\partial x^2} = -2\cdot 1\, y^3 z^4 k$$

$$\frac{\partial^2 \vec{F}}{\partial y^2} = -3\cdot 2\, x^2 y z^4 k$$

$$\frac{\partial^2 \vec{F}}{\partial z^2} = -4\cdot 3\, x^2 y^3 z^2 k$$

$$\nabla^2 \vec{F} = -\left(2 y^3 z^4 + 6 x^2 y z^4 + 12 x^2 y^3 z^2\right) k$$

$$\nabla^2 \vec{F} = \underline{-2 y z^2 \left(y^2 z^2 + 3 x^2 z^2 + 6 x^2 y^2\right) k}$$

(4) (a) $\nabla^2 f = \cancel{\frac{\partial^2}{\partial x^2}(e^z \sin y)} + \frac{\partial^2}{\partial y^2}(e^z \sin y) + \frac{\partial^2}{\partial z^2}(e^z \sin y)$

$$- e^z \sin y \quad + e^z \sin y \quad = 0 \qquad \underline{\text{yes}}$$

(b) $f = \sin x \sinh y + \cos x \cosh z$

$$\nabla^2 f = -\sin x\, \cancel{\sinh y} - \cancel{\cos x \cosh z} + \sin x\, \cancel{\sinh y} + 0 + 0 + \cancel{\cos x \cosh z} = 0$$

<u>yes</u>

(c) $f = \sin px \sinh qy$; $\nabla^2 f = -p^2 \sin px \sinh qy + q^2 \sin px \sinh qy = (-p^2 + q^2) \sin px \sinh$

<u>yes</u> only if $\underline{p^2 = q^2}$

3.6 p. 140 cont.

(5) a. ∇f vector field
b. $\nabla \cdot \vec{F}$ scalar
c. $\nabla \times \vec{F}$ vector
d. $\nabla \cdot \nabla f$ scalar $(\nabla^2 f)$
e. $\nabla \times \nabla f$ vector
f. $\nabla \times f$ meaningless
g. $\nabla^2 \vec{F}$ vector
h. $\nabla \times \underbrace{\nabla^2 \vec{F}}_{\text{vector}}$ vector
i. $\nabla \times \underbrace{(\nabla^2 f)}_{\text{scalar}}$ meaningless
j $\nabla (\nabla^2 f)$ vector

(7) $f = 2x^2 + y$
$\vec{R} = xi + yj + zk$

a. $\nabla f = 4x\,i + j$
b. $\nabla \cdot R = 1 + 1 + 1 = 3$
c. $\nabla^2 f = 4$
d. $$\nabla \times (fR) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x(2x^2+y) & y(2x^2+y) & z(2x^2+y) \end{vmatrix} = i(z-0) - j(4xz - 0) + k(4xy - x)$$

$$= \underline{zi - 4xzj + x(4y-1)k}$$

3.8 p. 150

(1) Prove $\nabla(\varphi_1\varphi_2) = \varphi_1\nabla\varphi_2 + \varphi_2\nabla\varphi_1$

$$\vec{i}\,\frac{\partial}{\partial x}(\varphi_1\varphi_2) + \vec{j}\,\frac{\partial}{\partial y}(\varphi_1\varphi_2) + k\,\frac{\partial}{\partial z}(\varphi_1\varphi_2)$$

using rule for derivative of product

$$\underbrace{i\left(\frac{\partial}{\partial x}\varphi_1\right)\varphi_2}_{\text{These will give } \varphi_2\nabla\varphi_2} + \underbrace{i\left(\frac{\partial}{\partial x}\varphi_2\right)\varphi_1}_{\text{These will give } \varphi_1\nabla\varphi_2} + \cdots$$

Prove $\nabla\cdot(\varphi F) = \varphi\,\nabla\cdot F + F\cdot\nabla\varphi$

$$\frac{\partial}{\partial x}(\varphi F_1) + \frac{\partial}{\partial y}(\varphi F_2) + \frac{\partial}{\partial z}(\varphi F_3)$$

$$= \underline{\left(\frac{\partial\varphi}{\partial x}\right)}F_1 + \varphi\frac{\partial F_1}{\partial x} + F_2\underline{\frac{\partial\varphi}{\partial y}} + \varphi\frac{\partial F_2}{\partial y} + F_3\underline{\frac{\partial\varphi}{\partial z}} + \varphi\frac{\partial F_3}{\partial z}$$

$$= F_1\frac{\partial\varphi}{\partial x} + F_2\frac{\partial\varphi}{\partial y} + F_3\frac{\partial\varphi}{\partial z} + \varphi\underbrace{\left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right]}$$

$$= F\cdot\nabla\varphi + \varphi\,(\nabla\cdot F)$$

3.8 p. 150 cont.

(5) $\nabla \cdot (F \times G) = G \cdot (\nabla \times F) + F \cdot (\nabla \times G)$ not valid

$F \times G \neq G \times F$ thus the left is not symmetric but the right is.

(6) $\nabla \cdot \dfrac{A \times R}{|R|} = 0$

$$A \times R = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = (a_2 z - y a_3) i - j (a_1 z - a_3 x) + k (a_1 y - a_2$$

$$\frac{A \times R}{|R|} = \frac{(a_2 z - a_3 y) i - j (a_1 z - a_3 x) + k (a_1 y - a_2 x)}{\sqrt{x^2 + y^2 + z^2}}$$

divergence of above $= \dfrac{\sqrt{x^2+y^2+z^2} \cdot 0 - \frac{1}{2}(x^2+y^2+z^2)^{-1/2} \cdot 2x (a_2 z - a_3 y}{(x^2+y^2+z^2)}$

$$+ \tfrac{1}{2} (x^2+y^2+z^2)^{-1/2} \, 2y (a_1 z - a_3 x)$$

$$- \tfrac{1}{2} (x^2+y^2+z^2)^{-1/2} \, 2z (a_1 y - a_2 x)$$

$$\frac{\text{Numerator}}{\text{denom.}} = \frac{-x(\cancel{a_2 z} - \cancel{a_3 y}) - y(\cancel{a_3 x} - \cancel{a_1 z}) - z(\cancel{a_1 y} - \cancel{a_2 x})}{(x^2+y^2+z^2)^{3/2}} = 0$$

3.8 p. 150 cont.

(10) a $\nabla\cdot(R^2\vec{A}) = 2x a_1 + 2y a_2 + 2z a_3 = 2\vec{R}\cdot\vec{A}$

b
$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ R^2 a_1 & R^2 a_2 & R^2 a_3 \end{vmatrix} = i(2y a_3 - 2z a_2) - j(2x a_3 - 2z a_1) + k(2x a_2 - 2y a_1)$$
$$= 2(\vec{R}\times\vec{A})$$

c $\vec{R}\cdot\nabla(R^2\vec{A}) = \vec{R}\cdot 2\vec{R}\cdot\vec{A} = 2R^2\vec{A}$

$2(\vec{R}\cdot\vec{R})\cdot\vec{A}$

d $\nabla(\vec{A}\cdot\vec{R})^4 = 4(\vec{A}\cdot\vec{R})^3\,\vec{A}$

e $\nabla\cdot(|\vec{R}|\vec{A}) = \dfrac{\vec{A}\cdot\vec{R}}{|\vec{R}|}$

f $\vec{R}\cdot\nabla(\vec{A}\cdot\vec{R}\vec{A}) = \vec{R}\cdot\underbrace{[\nabla(\vec{A}\cdot\vec{R})]}_{\vec{A}}\vec{A} = (\vec{R}\cdot\vec{A})\vec{A}$

g $\nabla\cdot(\vec{A}\times\vec{R}) = 0$ use (3.38)

$= \vec{R}\cdot\underbrace{\nabla\times\vec{A}}_{=0 \text{ since A is Const.}} - \vec{A}\cdot\underbrace{(\nabla\times\vec{R})}_{=0 \text{ by } 3.39}$

h $\nabla\times(\vec{A}\times\vec{R}) =$ use 3.39 $\underbrace{(\vec{R}\cdot\nabla)\vec{A}}_{=0 \text{ A is const.}} - (\vec{A}\cdot\nabla)\vec{R} + \underbrace{(\nabla\cdot\vec{R})}_{3 \text{ by } (3.31)}\vec{A} - \underbrace{(\nabla\cdot\vec{A})}_{=0}\vec{R}$

$= -\left(a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\right)\vec{R} + 3\vec{A}$

$= -a_1 i - j a_2 - k a_3 + 3\vec{A} = \underline{2\vec{A}}$

3.8 p. 150 cont.

(10) i $\quad \nabla^2(\vec{R}\cdot\vec{R}) = \nabla^2(x^2+y^2+z^2) = 6$

(12) $\quad V = (x+4y)\,i + (y-3z)\,j + Cz\,k = \nabla\times\vec{F}$

(3.43) $\nabla\cdot(\nabla\times\vec{F}) = 0 \Rightarrow \frac{\partial}{\partial x}(x+4y) + \frac{\partial}{\partial y}(y-3z) + \frac{\partial}{\partial z}(Cz) = 0$

$$1 + 1 + C = 0$$

$$\boxed{C = -2}$$

3.10 p. 169

(2) Use (3.50)–(3.51) to derive

$$e_z = k \; ; \quad e_\rho = \frac{x i + y j}{\sqrt{\quad}} \; ; \quad e_\theta = \frac{-y i + x j}{\sqrt{\quad}}$$

$$e_z = \frac{\text{grad}\, z}{|\text{grad}\, z|} \qquad \text{since } z = z$$

$$= \frac{k}{1} = k$$

$$e_\rho = \frac{\text{grad}\, \rho}{|\text{grad}\, \rho|} = \frac{\frac{\partial}{\partial x}\sqrt{\quad}\; i + \frac{\partial}{\partial y}\sqrt{\quad}\; j}{|\text{grad}\, \rho|}$$

$$\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2} \cdot \cancel{2}x = \frac{x}{\sqrt{\quad}}$$

$$\frac{\partial}{\partial y}\sqrt{x^2+y^2} = \frac{y}{\sqrt{x^2+y^2}}$$

$$|\text{grad}\, \rho|^2 = \left(\frac{x}{\sqrt{\quad}}\right)^2 + \left(\frac{y}{\sqrt{\quad}}\right)^2 = \frac{x^2+y^2}{x^2+y} = 1$$

$$\Rightarrow e_\rho = \frac{x i + y j}{\sqrt{x^2+y^2}}$$

Similarly for e_θ

$$\frac{\partial \sin^{-1}\frac{y}{\sqrt{\quad}}}{\partial x} = \frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}} \cdot \frac{0 - y\frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}\cancel{2}x}{x^2+y^2}$$

$$= \frac{\cancel{\sqrt{x^2+y^2}}}{x} \quad \frac{-xy\,\cancel{(x^2+y^2)^{-1/2}}}{x^2+y^2} = -\frac{y}{x(x^2+y^2)}$$

3.10' p. 169 cont.

2. cont

$$\frac{\partial}{\partial y}\left(\sin^{-1}\frac{y}{\sqrt{x^2+y^2}}\right) = \frac{1}{\sqrt{1-\frac{y^2}{x^2+y^2}}}\;\frac{\sqrt{} - y^2(x^2+y^2)^{-1/2}}{x^2+y^2}$$

$$= \frac{\sqrt{x^2+y^2}}{x}\;\frac{\sqrt{x^2+y^2} - y^2(x^2+y^2)^{-1/2}}{x^2+y^2}$$

$$= \frac{x^2+\cancel{y^2}-\cancel{y^2}}{x(x^2+y^2)} = \frac{x}{x^2+y^2}$$

$$|\nabla\theta|^2 = \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$$

$$e_\theta = \frac{\frac{-yi}{x^2+y^2} + \frac{xj}{x^2+y^2}}{\frac{1}{\sqrt{x^2+y^2}}} = \frac{-yi+xj}{\sqrt{x^2+y^2}}$$

3.10 p. 169 cont.

(4) Use (3.61) – (3.62) to derive

$$e_r = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$e_r = \frac{\nabla r}{|\nabla r|} = \frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{}} \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{}} \qquad \frac{\partial r}{\partial z} = \frac{z}{\sqrt{}}$$

$$|\nabla r|^2 = \frac{x^2+y^2+z^2}{(x^2+y^2+z^2)} = 1$$

Similarly for e_θ, e_φ

3.10 p. 169 cont.

(6) $\nabla^2 f$ in cylindrical

$$\nabla^2 f = \operatorname{div}(\operatorname{grad} f)$$

Let $$F = \nabla f = \underbrace{\frac{\partial f}{\partial \rho}}_{F_\rho} e_\rho + \underbrace{\frac{1}{\rho}\frac{\partial f}{\partial \theta}}_{F_\theta} e_\theta + \underbrace{\frac{\partial f}{\partial z}}_{F_z} e_z$$

$$\nabla^2 f = \operatorname{div} F = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial}{\partial \theta} F_\theta + \frac{\partial F_z}{\partial z}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial \theta}\left(\frac{1}{\rho}\frac{\partial f}{\partial \theta}\right) + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

In spherical coordinates:

$$\nabla f = \underbrace{\frac{\partial f}{\partial r}}_{F_r} e_r + \underbrace{\frac{1}{r}\frac{\partial f}{\partial \varphi}}_{F_\varphi} e_\varphi + \underbrace{\frac{1}{r\sin\varphi}\frac{\partial f}{\partial \theta}}_{F_\theta} e_\theta$$

$$\nabla \cdot F = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\varphi}\frac{\partial F_\theta}{\partial \theta} + \frac{1}{r\sin\varphi}\frac{\partial}{\partial \varphi}(\sin\varphi\, F_\varphi)$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\varphi}\frac{\partial}{\partial \theta}\left(\frac{1}{r\sin\varphi}\frac{\partial f}{\partial \theta}\right) +$$
$$\frac{1}{r\sin\varphi}\frac{\partial}{\partial \varphi}\left(\frac{\sin\varphi}{r}\frac{\partial f}{\partial \varphi}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi \frac{\partial f}{\partial \varphi}\right)$$

3.10 p.169 cont.

(8) a.

$$F = \frac{xi + yj}{x^2+y^2}$$

In cylindrical coordinates

$$F(\rho,\theta,z) = \frac{\rho e_\rho}{\rho^2} \qquad (\text{see } (5.4) \text{ for } xi+yj = \rho e_\rho)$$

$$\Rightarrow F_\rho = \frac{1}{\rho}, \quad F_\theta = F_z = 0$$

$$\nabla\cdot F = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho \underbrace{F_\rho}_{\frac{1}{\rho}}) = \frac{1}{\rho}\frac{\partial}{\partial\rho}(1) = 0$$

$$\nabla\times F = \frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ \frac{1}{\rho} & 0 & 0 \end{vmatrix} = 0$$

independent of θ, z

b
$$F = \frac{-yi+xj}{x^2+y^2} = \frac{e_\theta}{\rho} \qquad \text{see } (5.32)$$

$$F_\rho = 0 \quad F_\theta = \frac{1}{\rho} \quad F_z = 0$$

$$\nabla\cdot F = \frac{1}{\rho}\frac{\partial F_\theta}{\partial\theta} = 0$$

$$\nabla\times F = \frac{1}{\rho}\begin{vmatrix} e_\rho & \rho e_\theta & e_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & 1 & 0 \end{vmatrix} = 0$$

3.10 p. 169 cont.

11. $f(r,\theta,\varphi) = \dfrac{\cos\varphi}{r^2}$

$$\nabla f = \underbrace{\frac{\partial f}{\partial r}}_{-\frac{2\cos\varphi}{r^3}} e_r + \underbrace{\frac{\partial f}{\partial \theta}}_{=0} e_\theta + \underbrace{\frac{\partial f}{\partial \varphi}}_{-\frac{\sin\varphi}{r^2}} e_\varphi$$

$$\nabla f = -\frac{2\cos\varphi}{r^3} e_r - \frac{\sin\varphi}{r^2} e_\varphi$$

3.11 p.180

#3 since x, y, z are orthogonal
any order of x, y, z will do
same for ρ, θ, z

#4 $u_1 = e^x$

$u_2 = y$

$u_3 = z$

$$du_1\, du_2\, du_3 = e^x\, dx\, dy\, dz$$

Since $dV = dx\, dy\, dz = e^{-x}\, du_1\, du_2\, du_3$

$$= \frac{1}{u_1}\, du_1\, du_2\, du_3$$

$$\bar{e}_i = \frac{\bar{\nabla} u_i}{|\bar{\nabla} u_i|} = \frac{\frac{\partial \bar{x}}{\partial u_i}}{\left|\frac{\partial \bar{x}}{\partial u_i}\right|}$$

volume $= h_1 h_2 h_3 = e^x$

3.11 p.180 cont.

(5) $g = u_1^3 + u_2^3 + u_3^3$

$\nabla^2 g = ?$

$$\left.\begin{aligned} x &= u_1 - u_2 \\ y &= u_1 + u_2 \end{aligned}\right\} \Rightarrow \begin{aligned} u_1 &= \tfrac{1}{2}(x+y) \\ u_2 &= \tfrac{1}{2}(y-x) \\ u_3 &= \sqrt{z} \end{aligned}$$

$$z = u_3^2$$

$$g = \frac{1}{8}(x+y)^3 + \frac{1}{8}(y-x)^3 + z^{3/2}$$

$$\nabla^2 g = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2} = \underbrace{\frac{3}{4}(x+y) + \frac{3}{4}(y-x)}_{=\frac{\partial^2 g}{\partial x^2}} + \underbrace{\frac{3}{4}(x+y) + \frac{3}{4}(y-x)}_{=\frac{\partial^2 g}{\partial y^2}} + \left(\frac{3}{4} z^{-1/2}\right)$$

$$= \frac{3}{2}(x+y) + \frac{3}{2}(y-x) + \frac{3}{4}\frac{1}{\sqrt{z}}$$

in terms of u_i:

$$\boxed{\nabla^2 g = 3u_1 + 3u_2 + \frac{3}{4u_3}}$$

Or when using Laplacian on p.179

$h_1 = \sqrt{2} \quad h_2 = \sqrt{2} \quad h_3 = 2u_3 \qquad (3.80)$

$$\nabla^2 f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$$

$$= \frac{1}{4u_3}\left[\frac{\partial}{\partial u_1}\left(\frac{2\cancel{\sqrt{2}}u_3}{\cancel{\sqrt{2}}} 3u_1^2\right) + \frac{\partial}{\partial u_2}\left(2u_3 \cdot 3u_2^2\right) + \frac{\partial}{\partial u_3}\left(\frac{\cancel{2}}{\cancel{2}\cancel{u_3}} 3u_3^{\cancel{2}}\right)\right]$$

$$= \frac{1}{4u_3}\left[12u_1u_3 + 12u_2u_3 + 3\right]$$

$$= 3u_1 + 3u_2 + \frac{3}{4u_3}$$

4.3 cont. p. 204

6. $F = (y + z\cos xz)i + xj + x\cos xz\, k$

$$\frac{\partial\varphi}{\partial x} = y + z\cos xz$$

$$\frac{\partial\varphi}{\partial y} = x \Longrightarrow \varphi = xy + p(x,z)$$

$$\frac{\partial\varphi}{\partial z} = x\cos xz$$

$$\Downarrow$$

$$\frac{\partial\varphi}{\partial x} = y + \frac{\partial p(x,z)}{\partial x} = y + z\cos xz$$

$$\frac{\partial\varphi}{\partial z} = \frac{\partial p(x,z)}{\partial z} = x\cos xz \qquad (*)$$

$$\frac{\partial p}{\partial x} = z\cos xz$$

$$\Downarrow$$

$$p = \sin(xz) + g(z) \qquad (\#)$$

$$\Downarrow$$

$$\frac{\partial p}{\partial z} = x\cos xz + g'(z)$$

but from $(*)$ $\frac{\partial p}{\partial z} = x\cos xz$

$$\Downarrow$$

$$g'(z) = 0$$

$$g(z) = C$$

$\Downarrow$ subst. in(#)

$$p = \sin xz + C$$

$$\Downarrow$$

$$\boxed{\varphi = xy + \sin xz}$$

3.11 p. 180 cont.

#8 $x = u_1^2 - u_2^2$

$y = 2u_1 u_2$ $\qquad R = (u_1^2 - u_2^2)i + 2u_1u_2 j + u_3 k$

$z = u_3$

a.

$$e_1 = \frac{\nabla u_1}{|\nabla u_1|} = \frac{\frac{\partial R}{\partial u_1}}{\left|\frac{\partial R}{\partial u_1}\right|} = \frac{2u_1 i + 2u_2 j}{\sqrt{4u_1^2 + 4u_2^2}} = \frac{u_1 i + u_2 j}{\sqrt{u_1^2 + u_2^2}}$$

$$e_2 = \frac{\frac{\partial R}{\partial u_2}}{\left|\frac{\partial R}{\partial u_2}\right|} = \frac{-2u_2 i + 2u_1 j}{\sqrt{4u_1^2 + 4u_2^2}} = \frac{-u_2 i + u_1 j}{\sqrt{u_1^2 + u_2^2}}$$

$$e_3 = \frac{k}{1} = k$$

orthogonality : $e_1 \cdot e_2 = \dfrac{-u_1 u_2 + u_1 u_2}{u_1^2 + u_2^2} = 0$

$e_2 \cdot e_3 = e_1 \cdot e_3 = 0$ $\square$

b. scale factors : $h_1 = h_2 = 2\sqrt{u_1^2 + u_2^2}$

$h_3 = 1$

c $$\nabla^2 g = \frac{1}{4(u_1^2+u_2^2)}\left[\frac{\partial^2 g}{\partial u_1^2} + \frac{\partial^2 g}{\partial u_2^2} + \frac{\partial}{\partial u_3}\left(4(u_1^2+u_2^2)g\right)\right]$$

$$= \frac{1}{4(u_1^2+u_2^2)}\left(\frac{\partial^2 g}{\partial u_1^2} + \frac{\partial^2 g}{\partial u_2^2}\right) + \frac{\partial^2 g}{\partial u_3^2}$$

d $F = u_3 \underline{e}_1 + u_1 \underline{e}_2 + u_2 \underline{e}_3$

$$\nabla \cdot F = \frac{1}{4(u_1^2+u_2^2)}\left[\frac{\partial}{\partial u_1}\left(2\sqrt{u_1^2+u_2^2}\, u_3\right) + \frac{\partial}{\partial u_2}\left(2\sqrt{u_1^2+u_2^2}\, u_1\right) + \underbrace{\frac{\partial}{\partial u_3}\left(4(u_1^2+u_2^2)u_2\right)}_{=0}\right]$$

3.11 p.180 cont.

3d cont

$$\nabla \cdot F = \frac{1}{\overset{\cancel{4}}{2}(u_1^2+u_2^2)}\left\{\cancel{2}u_3 \tfrac{1}{\cancel{2}}(u_1^2+u_2^2)^{-1/2}\cdot \cancel{2}u_1 + \cancel{2}u_1 \tfrac{1}{\cancel{2}}(u_1^2+u_2^2)^{-1/2}\cancel{2}u_2\right\}$$

$$= \frac{u_1 u_3 + u_1 u_2}{2\,(u_1^2+u_2^2)^{3/2}}$$

$$\text{curl } F = \frac{1}{4(u_1^2+u_2^2)}\begin{vmatrix} 2\sqrt{u_1^2+u_2^2}\, e_1 & 2\sqrt{u_1^2+u_2^2}\, e_2 & e_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ 2u_3\sqrt{u_1^2+u_2^2} & 2u_1\sqrt{u_1^2+u_2^2} & u_2 \end{vmatrix}$$

$$= \frac{1}{4(u_1^2+u_2^2)}\Big\{2\sqrt{u_1^2+u_2^2}\, e_1(1-0) - 2\sqrt{u_1^2+u_2^2}\, e_2\left(0 - 2\sqrt{u_1^2+u_2^2}\right)$$

$$+ e_3\left(2\sqrt{u_1^2+u_2^2} + 2u_1\frac{\cancel{2}u_1}{\cancel{2}\sqrt{u_1^2+u_2^2}} - 2u_3\frac{\cancel{2}u_2}{\cancel{2}\sqrt{u_1^2+u_2^2}}\right)\Big\}$$

$$= \frac{1}{2\sqrt{u_1^2+u_2^2}}e_1 + e_2 + e_3\left(\frac{1}{2\sqrt{u_1^2+u_2^2}} + \frac{u_1^2}{2(u_1^2+u_2^2)^{3/2}} - \frac{u_2 u_3}{2(u_1^2+u_2^2)^{3/2}}\right)$$

3.11 p. 180 cont.

9. $x = u_3$

$y = e^{u_2}\cos u_1$

$z = e^{u_2}\sin u_1$

a. $\frac{\partial R}{\partial u_1} = -\sin u_1\, e^{u_2} j + \cos u_1\, e^{u_2} k$

$\frac{\partial R}{\partial u_2} = e^{u_2}\cos u_1\, j + e^{u_2}\sin u_1\, k$

$\frac{\partial R}{\partial u_3} = i$

$\frac{\partial R}{\partial u_1} \perp \frac{\partial R}{\partial u_3}$ & $\frac{\partial R}{\partial u_2} \perp \frac{\partial R}{\partial u_3}$ since $\frac{\partial R}{\partial u_3}$ have only $\vec{i}$ component and the others do not.

Is $\frac{\partial R}{\partial u_1} \perp \frac{\partial R}{\partial u_2}$?

$\frac{\partial R}{\partial u_1} \cdot \frac{\partial R}{\partial u_2} = -\sin u_1 \cos u_1\, e^{2u_2} + \sin u_1 \cos u_1\, e^{2u_2} = 0$ ✓

b. $h_1 = \left|\frac{\partial R}{\partial u_1}\right| = \left| -\sin u_1\, e^{u_2} j + \cos u_1\, e^{u_2} k \right|$

$= \underline{e^{u_2}}$

$h_2 = \underline{e^{u_2}}$ same way

$h_3 = \underline{1}$

3.11 p.180 cont.

9c $\nabla^2 (u_1^2 + u_2^2 + u_3^2) = ?$

$$\nabla^2 g = \frac{1}{e^{2u_2}} \left[\frac{\partial}{\partial u_1}\left(\frac{\partial g}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{\partial g}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(e^{2u_2}\frac{\partial g}{\partial u_3}\right)\right]$$

$$= e^{-2u_2}\left[2 \quad + \quad 2 \quad + \quad 2e^{2u_2}\right]$$

$$= \underline{4e^{-2u_2} + 2}$$

d $\nabla \cdot (-e^{u_2} \underline{e}_3 + u_3 \underline{e}_1)$

$$\nabla \cdot F = \frac{1}{e^{2u_2}} \left[\underbrace{\frac{\partial}{\partial u_1}(u_3 e^{u_2})}_{=0} + \underbrace{\frac{\partial}{\partial u_2}(0)}_{=0} + \underbrace{\frac{\partial}{\partial u_3}(-e^{u_2}e^{2u_2})}_{=0}\right] = 0$$

$$\nabla \times F = \frac{1}{e^{2u_2}} \begin{vmatrix} e^{u_2} e_1 & e^{u_2} e_2 & e_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ e^{u_2} u_3 & \cancel{e^{u_2}}\ 0 & -e^{2u_2} \end{vmatrix}$$

$$= \frac{1}{e^{2u_2}} \left\{ e^{u_2} \underline{e}_1 (-2e^{2u_2}) - e^{u_2} \underline{e}_2 (-e^{u_2}) + \underline{e}_3 e^{u_2} u_3 \right\}$$

$$= \frac{1}{e^{2u_2}} \left\{ -2e^{3u_2} \underline{e}_1 + e^{2u_2} \underline{e}_2 + e^{u_2} u_3 \underline{e}_3 \right\}$$

3.11 p. 180 cont.

12. $x = \frac{1}{2}(u^2 - v^2)$ $\quad -\infty < u < \infty$

$y = uv$ $\quad v \geq 0$

$z = z$ $\quad -\infty < z < \infty$

parabolic cylindrical coordinates

$h_u = \left|\frac{\partial R}{\partial u}\right| = |u\,i + v\,j| = \sqrt{u^2 + v^2}$

$h_v = |-v\,i + u\,j| = \sqrt{v^2 + u^2}$

$h_z = 1$

13. $dV = h_1 h_2 h_3 \, du_1 \, du_2 \, du_3$

In parabolic cylindrical:

$dV = (u^2 + v^2)\, du\, dv\, dz$

4.1. p.190

(1) $C: x = t$ $\quad F = x^2 i + j + yz k$

$y = 2t^2 \qquad 0 \le t \le 1$

$z = 3t$

$\Rightarrow dx = dt$

$dy = 4t\,dt$

$dz = 3\,dt$

$$\int_0^1 x^2\,dx + dy + yz\,dz$$

$$= \int_0^1 t^2\,dt + 4t\,dt + 2t^2 \cdot 3t \cdot 3\,dt$$

$$= \int_0^1 (t^2 + 4t + 18t^3)\,dt = \frac{1}{3}t^3 + 2t^2 + 18\frac{t^4}{4}\Big|_0^1$$

$$= \frac{1}{3} + 2 + \frac{9}{2} - 0 = \frac{2+12+27}{6} = \underline{\underline{\frac{41}{6}}}$$

(3) $\int (xi + yj + zk)\cdot dR$ from $(1,0,0)$ to $(1,0,4)$

a. along line joining points

$x = 1$

$y = 0 \qquad 0 \le t \le 1 \qquad \Rightarrow \qquad dx = dy = 0 \quad dz = 4\,dt$

$z = 4t$

$$\text{Integ.} = \int_0^1 4t \cdot 4\,dt = 16\,\frac{t^2}{2}\Big|_0^1 = \underline{\underline{8}}$$

b along $x = \cos 2\pi t$ $\quad dx = -2\pi \sin 2\pi t\,dt$

$y = \sin 2\pi t \quad dy = 2\pi \cos 2\pi t$

$z = 4t \quad dz = 4\,dt$

$0 \le t \le 1$

$$\int_0^1 \Big[\underbrace{\cancel{\cos 2\pi t\,(-2\pi \sin 2\pi t)} - \sin 2\pi t\,\cancel{(2\pi \cos 2\pi t)}}_{= -4\pi \sin 2\pi t \cos 2\pi t \,=\, -2\pi \sin 4\pi t} + 4t\cdot 4\Big]dt$$

The terms in the integrand add up to $-2\pi \sin 4\pi t$

$$= -2\pi \int_0^1 \sin 4\pi t\,dt = \cancel{2\pi}\,\frac{\cos 4\pi t}{\cancel{4\pi}_{\,2}}\Big|_0^1 = \frac{1}{2}(\underbrace{\cos 4\pi}_{=1} - 1) = 0$$

4.1 p.190 cont.

(6) $\oint F \cdot dR$

$$\left.\begin{array}{l} x^2 - 2x + y^2 = 2 \\ z = 1 \end{array}\right\} \text{circle}$$

$$(x-1)^2 + y^2 = 3$$
$$\Downarrow$$
$$x = \sqrt{3}\cos\theta + 1$$
$$y = \sqrt{3}\sin\theta \qquad 0 \le \theta \le 2\pi$$

$$F = y\,i + x\,j + xyz^2 k$$

$$\int_0^{2\pi} \Big[\sqrt{3}\sin\theta \cdot (-\sqrt{3}\sin\theta) + (\sqrt{3}\cos\theta + 1)(\sqrt{3}\cos\theta) + \cancel{(\sqrt{3}\cos\theta+1)\sqrt{3}\sin\theta \cdot 1^2 \cdot 0}\Big]\, d\theta$$

z fixed

$$\int_0^{2\pi} \big(\underbrace{-3\sin^2\theta + 3\cos^2\theta}_{+3\cos 2\theta} + \sqrt{3}\cos\theta\big)\, d\theta = \frac{3}{2}\sin 2\theta + \sqrt{3}\sin\theta \Big|_0^{2\pi}$$

$$= \frac{3}{2}\sin 4\pi + \sqrt{3}\sin 2\pi - \cdots = \underline{\underline{0}}$$

all zeros.

(10)
$$\int_0^{2\pi} \big(3\sin\theta + 3\sin\theta \cdot 1\, \cancel{\cos(6\sin\theta\cos\theta)}\big)\, \underbrace{(-2\sin\theta)\, d\theta}_{dx}$$
$$+ \int_0^{2\pi} \big(4\cos^2\theta + \cancel{2\cos\theta\cos(6\sin\theta\cos\theta)}\big)\, \underbrace{3\cos\theta\, d\theta}_{dy}$$
$$+ \int_0^{2\pi} \cancel{(1 + 6\sin\theta\cos\theta\ \cos(6\sin\theta\cos\theta))\, \underbrace{0\, d\theta}_{dz}}$$

Last integral vanishes since z is a constant.

Now let's show that the two terms crossed out will vanish

$$\int_0^{2\pi} \Big[-6\sin^2\theta\cos(\underbrace{6\sin\theta\cos\theta}_{3\sin 2\theta}) + 6\cos^2\theta\cos(6\sin\theta\cos\theta)\Big]\, d\theta$$

$$= \int_0^{2\pi} \cos(3\sin 2\theta)\,\big[\underbrace{6\cos^2\theta - 6\sin^2\theta}_{6\cos 2\theta}\big]\, d\theta$$

Substitute $u = 3\sin 2\theta \Rightarrow du = 6\cos 2\theta\, d\theta$

$$= \int \cos u\, du = +\sin u = \sin(3\sin 2\theta)\Big|_{\theta=0}^{\theta=2\pi} = 0 - 0.$$

4.1 p.190 cont.

10. cont.

So we have left with

$$\int_0^{2\pi} \left(-6\sin^2\theta \quad + 12\underbrace{\cos^3\theta}_{(1-\sin^2\theta)\cos\theta}\right) d\theta$$

$$= -3\theta + 3\cdot\tfrac{1}{2}\sin 2\theta + 12\sin\theta\Big|_0^{2\pi} - 12\int_0^{2\pi}\sin^2\theta\cos\theta\, d\theta$$

$u = \sin\theta \quad du = \cos\theta\, d\theta$

$$= -6\pi - 4\sin^3\theta\Big|_0^{2\pi} = \boxed{-6\pi}$$

(14) a

$$F = \underset{\substack{\uparrow \\ \text{const.}}}{\omega} \times R \quad \Rightarrow \quad F \perp R \quad \Rightarrow \quad F\cdot dR = 0$$

$$\int F\cdot dR = \int(\omega\times R)\cdot dR = 0$$

(18) $F = \frac{x^2}{y} i + y j + k$

(a) flow line for F thru $(1,1,0)$

$F_1 = \frac{x^2}{y}$ $\quad F_2 = y$ $\quad F_3 = 1$

$$\frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} \Rightarrow \underbrace{\frac{y\,dx}{x^2} = \frac{dy}{y}} = \frac{dz}{1}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} \qquad\qquad \frac{dy}{y} = dz$$

$$-\frac{1}{x} = -\frac{1}{y} - C \qquad\qquad \ln y = z + C$$

at $(1,1,0)$ $\qquad\qquad$ at $(1,1,0)$

$$-1 = -1 - C \qquad\qquad \ln 1 = 0 + C$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$C = 0 \qquad\qquad C = 0$$

$$\Rightarrow \boxed{y = x} \qquad\qquad \boxed{\ln y = z}$$

$$\Rightarrow \boxed{x = t \quad y = t \quad z = \ln t}$$

(b) at $(e, e, 1)$ then $t = e$ $\Rightarrow x = e \quad y = e \quad z = \ln e = 1$ which is on flow line

(c) $$\int_{(1,1,0)\text{ flow line}}^{(e,e,1)} \left(\frac{x^2}{y} i + yj + k\right) \cdot dR$$

$$= \int \left(\frac{x^2}{y} dx + y\,dy + .\ dz\right)$$

$x = t \quad y = t \quad z = \ln t$ is the flow line

$\Rightarrow \quad dx = dt \quad dy = dt \quad dz = \frac{1}{t} dt$

$$= \int_{t=1}^{t=e} \left(\frac{t^2}{t} dt + t\,dt + \frac{1}{t} dt\right)$$

$$= \frac{1}{2}t^2\Big|_1^e + \frac{1}{2}t^2\Big|_1^e + \ln t\Big|_1^e$$

$$= \frac{1}{2}e^2 - \cancel{\frac{1}{2}} + \frac{1}{2}e^2 - \cancel{\frac{1}{2}} + \cancel{1} - 0 = \underline{\underline{e^2}}$$

4.3 p. 204.

non conservative

(2) a $F = -y\,i + x\,j$

$\frac{\partial \varphi}{\partial x} = -y \qquad \frac{\partial \varphi}{\partial y} = x$

$\Downarrow$

$\varphi = -yx + p(y) \qquad -x + p'(y) = x$

$p'(y) = 2x$

$\uparrow$

not possible to have a function of y only to have x on the right

b $F = y\,i + y(x-1)\,j$

$\frac{\partial \varphi}{\partial x} = y \qquad \varphi = xy + p(y)$

$\Downarrow$

$\frac{\partial \varphi}{\partial y} = y(x-1) = x + p'(y)$

$yx - y - x = p'(y)$ same as before

c $F = y\,i + x\,j + x^2 k$

$\frac{\partial \varphi}{\partial x} = y \qquad \varphi = xy + p(y,z)$

$\frac{\partial \varphi}{\partial y} = x \qquad \cancel{x} = \cancel{x} + \frac{\partial p}{\partial y} \Rightarrow p = g(z) \Rightarrow \varphi = xy + g(z)$

$\frac{\partial \varphi}{\partial z} = x^2 \qquad x^2 = g'(z)$ same as before.

d $F = z\,i + z\,j + (y-1)\,k$

$\frac{\partial \varphi}{\partial x} = z \qquad \varphi = xz + p(y,z)$

$\frac{\partial \varphi}{\partial y} = z = \frac{\partial p}{\partial y} \Rightarrow p = yz + g(z) \Rightarrow \varphi = xz + yz + g(z)$

$\frac{\partial \varphi}{\partial z} = y - 1 \qquad \cancel{y} - 1 = x + \cancel{y} + g'(z)$ same as before

4.3 cont. p. 204

2e $F = \frac{x}{x^2+y^2} i + \frac{x}{x^2+y^2} j$

$$\frac{\partial\varphi}{\partial x} = \frac{x}{x^2+y^2}$$

$$\frac{\partial\varphi}{\partial y} = \frac{x}{x^2+y^2} \qquad \varphi = \arctan\frac{y}{x} + p(x)$$

$$\Downarrow$$

$$\frac{\partial\varphi}{\partial x} = \frac{-\frac{y}{x^2}}{1+\left(\frac{y}{x}\right)^2} + p'(x)$$

$$\frac{-y}{x^2+y^2} + p'(x) = \frac{x}{x^2+y^2}$$

$$p'(x) = \frac{x+y}{x^2+y^2} \quad \text{not possible}$$

(4) $\oint_C F\cdot dR \qquad C: x^2+y^2=r^2$

$$F = \frac{-y i + x j}{x^2+y^2}$$

polar : $x = r\cos\theta \qquad dx = -r\sin\theta\, d\theta$

$y = r\sin\theta \qquad dy = r\cos\theta\, d\theta$

$$\int_0^{2\pi}\left[\frac{-\cancel{r}\sin\theta}{\cancel{r^2}}(-\cancel{r}\sin\theta\, d\theta) + \frac{\cancel{r}\cos\theta}{\cancel{r^2}}(\cancel{r}\cos\theta\, d\theta)\right]$$

$$\int_0^{2\pi} d\theta = \underline{\underline{2\pi}}$$

(5) $\arctan\frac{y}{x}$ is multiple valued

4.3 cont. p. 204

6. $F = (y + z\cos xz)\,i + x\,j + x\cos xz\,k$

$$\frac{\partial \varphi}{\partial x} = y + z\cos xz$$

$$\frac{\partial \varphi}{\partial y} = x \Longrightarrow \varphi = xy + p(x,z)$$

$$\frac{\partial \varphi}{\partial z} = x\cos xz$$

$$\Downarrow$$

$$\frac{\partial \varphi}{\partial x} = y + \frac{\partial p(x,z)}{\partial x} = y + z\cos xz$$

$$\frac{\partial \varphi}{\partial z} = \frac{\partial p(x,z)}{\partial z} = x\cos xz \qquad (*)$$

$$\frac{\partial p}{\partial x} = z\cos xz$$

$$\Downarrow$$

$$p = \sin(xz) + g(z) \qquad (\#)$$

$$\Downarrow$$

$$\frac{\partial p}{\partial z} = x\cos xz + g'(z)$$

but from $(*)$ $\quad \frac{\partial p}{\partial z} = x\cos xz$

$$\Downarrow$$

$$g'(z) = 0$$

$$g(z) = C$$

$\Downarrow$ subst. in(#)

$$p = \sin xz + C$$

$$\Downarrow$$

$$\boxed{\varphi = xy + \sin xz}$$

4.3 cont. p. 204

(7) $F = 2xy\, i + (x^2+z)\, j + y\, k$ is conservative

$$\frac{\partial \varphi}{\partial x} = 2xy \Rightarrow \varphi = x^2 y + p(y,z)$$

$$\frac{\partial \varphi}{\partial y} = x^2 + z \qquad \cancel{x^2} + \frac{\partial p}{\partial y} = \cancel{x^2} + z$$

$$p = yz + g(z)$$

$$\frac{\partial \varphi}{\partial z} = y \qquad \varphi = x^2 y + yz + g(z)$$

$$\frac{\partial \varphi}{\partial z} = y = y + g'(z)$$

$$g'(z) = 0$$

$$g(z) = C$$

$$\underline{\varphi = x^2 y + yz + C}$$

4.4 p. 212

1

a. $F = (12xy + yz)i + (6x^2 + xz)j + xyk$

$\nabla \times F \overset{?}{=} 0$

$12x + z \overset{?}{=} 12x + z$

$x = x$

$y = y$

yes.

b. $F = ze^{xz} i + xe^{xz} k$

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = 0 \qquad (F_2 = 0)$$

$$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} = 0$$

$$\frac{\partial F_1}{\partial z} = xze^{xz} \overset{?}{=} \frac{\partial F_3}{\partial x} = xze^{xz}$$

yes

c. $\sin x \; i + y^2 j + e^z k$

$0 = 0$

$0 = 0$

$0 = 0$ yes

d. $F = 3x^2yz^2 i + x^3z^2 j + x^3yz k$

$3x^2z^2 = 3x^2z^2$

$2x^3z = x^3z$ no!

4.4 cont. p.212

e. $F = \frac{2x}{x^2+y^2} i + \frac{2y}{x^2+y^2} j + 2zk$

Test not application $x^2+y^2 \neq 0$

We can try to find the potential

1. $\frac{\partial \varphi}{\partial x} = \frac{2x}{x^2+y^2}$ $\Rightarrow$ $\varphi = \frac{1}{y^2}\int \frac{2x}{\left(\frac{x}{y}\right)^2+1}dx = \int \frac{du}{u} = \ln|u| + p(y,z)$

$u = \left(\frac{x}{y}\right)^2 + 1 \Rightarrow du = 2\left(\frac{x}{y}\right)\frac{1}{y}\,dx$

2. $\frac{\partial \varphi}{\partial y} = \frac{2y}{x^2+y^2}$

$$\varphi = \ln\left|\underbrace{1+\frac{x^2}{y^2}}\right| + p(y,z)$$

3. $\frac{\partial \varphi}{\partial z} = 2z$

>0 always and we can remove the abs. value.

4. $\varphi = \ln\left(1+\frac{x^2}{y^2}\right) + p(y,z)$

$$\frac{\partial \varphi}{\partial y} = \frac{1}{1+\frac{x^2}{y^2}}\left(-2x^2y^{-3}\right) + \frac{\partial p}{\partial y}$$

$$\frac{\partial \varphi}{\partial y} = \frac{-2x^2}{y^3\left(1+\frac{x^2}{y^2}\right)} + \frac{\partial p}{\partial y} = \frac{2y}{x^2+y^2}$$

from 2.

$$\Rightarrow \frac{\partial p}{\partial y} = \frac{2y}{x^2+y^2} + \frac{2x^2}{y(x^2+y^2)}$$

$$= \frac{2y^2+2x^2}{y(x^2+y^2)} = \frac{2}{y}$$

$$\Rightarrow p(y,z) = 2\ln|y| + g(z)$$

use 4 $\Rightarrow \varphi = \ln\left(1+\frac{x^2}{y^2}\right) + 2\ln|y| + g(z)$

$\underset{\text{from } 3}{2z} = \frac{\partial \varphi}{\partial z} = 0 + 0 + g'(z) \Rightarrow g = z^2 + C$

$$\Rightarrow \varphi = \ln\left(1+\frac{x^2}{y^2}\right) + \underbrace{2\ln|y|}_{\ln y^2} + z^2$$

$\varphi = \ln\left[y^2\left(1+\frac{x^2}{y^2}\right)\right] + z^2$

$\varphi = \ln(x^2+y^2) + z^2$

answer

4.4, cont. p. 212

(2)

$F = \nabla \varphi$
$G = \nabla \psi$

Is $F+G$ conservative ?

yes

$F+G = \nabla(\varphi+\psi)$

$\varphi+\psi$ is the potential

(6) a $F = (6x - 2e^{2x}y^2)\,i - 2ye^{2x}\,j + \cos z\,k$

$\frac{\partial\varphi}{\partial x} = 6x - 2e^{2x}y^2$ $\qquad \varphi = 3x^2 - e^{2x}y^2 + g(y,z)$

$\frac{\partial\varphi}{\partial y} = -2ye^{2x}$ $\qquad \frac{\partial\varphi}{\partial y} = -2e^{2x}y + \frac{\partial g}{\partial y}$

$\Rightarrow \frac{\partial g}{\partial y} = 0 \Rightarrow g = h(z)$

$\varphi = 3x^2 - e^{2x}y^2 + h(z)$

$\frac{\partial\varphi}{\partial z} = \cos z$ $\qquad \frac{\partial\varphi}{\partial z} = h'(z)$

$\Rightarrow h(z) = \sin z$

$\boxed{\varphi = 3x^2 - e^{2x}y^2 + \sin z + \cancel{\text{constant}}}$

conservative

b $R = \underbrace{t}_{x}\,i + \underbrace{(t-1)(t-2)}_{y}\,j + \underbrace{\frac{\pi}{2}t^3}_{z}\,k \qquad 0 \le t \le 1$

Since F is conservative (part a) we use the potential φ. When $t=0$ $\quad x=0,\ y=-1\cdot(-2)=2 \quad z=0$

$t=1 \quad x=1,\ y=0,\ z=\frac{\pi}{2}$

$\varphi(t=1) = \varphi(0,2,0) = 0 - 4 + 0 = -4$

$\varphi(t=0) = \varphi(1,0,\pi/2) = 3+1 = 4 \qquad \Rightarrow$ The integral $= -4-4 = \underline{\underline{-8}}$

4.4 cont. p. 212

6c. $R(t) = \underbrace{\tfrac{1}{2}(t-1)}_{x}\, i + \underbrace{t(3-t)}_{y}\, j + \underbrace{\tfrac{\pi}{4}(t-1)}_{z}\, k \qquad 1 \le t \le 3$

$t=1 \quad x=0 \quad y=2 \quad z=0$

$t=3 \quad x=1 \quad y=0 \quad z=\pi/2$

$\varphi(t=1) = \varphi(0,2,0) = -4$

$\varphi(t=3) = \varphi(1,0,\pi/2) = 4$

The integral $= 4-(-4) = \underline{\underline{8}}$

4.4 cont. p.212

7. $F = (1+x)e^{x+y}\, i + (x e^{x+y} + 2y)\, j - 2z k$

$\frac{\partial \varphi}{\partial x} = (1+x)e^{x+y}$

$$\varphi = e^{x+y} + e^{y} \underbrace{\int x e^{x} dx}_{e^{x}(x-1)} + g(y,z)$$

$$\varphi = xe^{x+y} + g(y,z)$$

$$\frac{\partial \varphi}{\partial y} = xe^{x+y} + \frac{\partial g}{\partial y} = xe^{x+y} + 2y$$

$$\frac{\partial g}{\partial y} = 2y$$

$$g = y^2 + h(z)$$

$$\varphi = xe^{x+y} + y^2 + h(z)$$

$$\frac{\partial \varphi}{\partial z} = h'(z) = -2z$$

$$h(z) = -z^2$$

b

$$\underline{\varphi = xe^{x+y} + y^2 - z^2}$$

$$\int_C (1+x)e^{x+y} dx + (x e^{x+y} + 2z)\, dy - 2y\, dz$$

$x = (1-t)e^{t}$ $dx = (-e^{t} + (1-t)e^{t})\, dt$

$y = t$ $0 \le t \le 1$ $dy = dt$

$z = 2t$ $dz = 2dt$

It's messy to do it straight, but using the fact the F is conservative and similarity of F, G

$G = F + H$ where $H = 2(z-y)\, j + 2(z-y)\, k$

$$\int_C G \cdot dR = \int_C F \cdot dR + \int_C H \cdot dR$$

4.4 cont. p. 212

7b cont.

$$\int_C F\cdot dR = \varphi(t=1) - \varphi(t=0)$$

$t=1 \Rightarrow x=0,\ y=1,\ z=2$
$t=0 \Rightarrow x=1,\ y=0,\ z=0$

$$= (1-4) - e = \underline{-3-e}$$

$$\int_C H\cdot dR = \int 2(z-y)\,dy + 2(z-y)\,dz$$

$$= \int_0^1 \underbrace{2(2t-t)}_{2t}\,dt + \underbrace{2(2t-t)\cdot 2}_{4t}\,dt$$

$$= \int_0^1 6t\,dt = 3t^2\Big|_0^1 = 3$$

$$\int_C G\cdot dR = -3-e+3 = \underline{\underline{-e}}$$

9. a $F = (2xyz + z^2 - 2y^2 + 1)i + (x^2z - 4xy)j + (x^2y + 2xz - 2)k$

$$\frac{\partial\varphi}{\partial x} = 2xyz + z^2 - 2y^2 + 1 \qquad \varphi = x^2yz + (z^2 - 2y^2 + 1)x + g(y,z)$$

$$\frac{\partial\varphi}{\partial y} = x^2z - 4yx + \frac{\partial g}{\partial y} = x^2z - 4xy$$

$$\frac{\partial g}{\partial y} = 0$$

$$g = h(z)$$

$$\varphi = x^2yz + (z^2 - 2y^2 + 1)x + h(z)$$

$$\frac{\partial\varphi}{\partial z} = x^2y + 2zx + h' = x^2y + 2xz - 2$$

$$h' = -2$$

$$h = -2z$$

$$\underline{\varphi = x^2yz + (z^2 - 2y^2 + 1)x - 2z}$$

4.4 cont. p. 212

9b. $$G = \frac{x}{(x^2+z^2)^2}\, i + \frac{z}{(x^2+z^2)^2}\, k$$

Is G. conservative?

$\nabla \times G = 0$ except of y axis ($x^2+z^2=0$ then)

G is not conservative since the domain is not star shaped.

<u>4.4 cont. p.212</u>

<u>10</u>

$F = (15x^4 - 3x^2y^2)\,i + (-2x^3y)\,j$

$$\frac{\partial \varphi}{\partial x} = 15x^4 - 3x^2y^2$$

$$\varphi = 3x^5 - x^3y^2 + g(y)$$

$$\frac{\partial \varphi}{\partial y} = -2x^3y + g' = -2x^3y$$

$$g' = 0 \quad \Rightarrow \quad g = \text{const.}$$

$$\varphi = 3x^5 - x^3y^2$$

F is conservative

$$\int_{(0,0)}^{(1,2)} F \cdot dR = \varphi(1,2) - \varphi(0,0) = (3-4) - 0 = \underline{\underline{-1}}$$

If we didn't go this route the solution is messy

4.6 p. 236

① S is given by $x = u^2$
$y = \sqrt{2}\, uv$
$z = v^2$

$$d\vec{S} = \frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v}\, du\, dv$$

$$\frac{\partial R}{\partial u} = 2u\, i + \sqrt{2}\, v\, j$$

$$\frac{\partial R}{\partial v} = \sqrt{2}\, u\, j + 2v\, k$$

$$\frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v} = \begin{vmatrix} i & j & k \\ 2u & \sqrt{2}v & 0 \\ 0 & \sqrt{2}u & 2v \end{vmatrix} = i \begin{vmatrix} \sqrt{2}v & 0 \\ \sqrt{2}u & 2v \end{vmatrix} - 2u \begin{vmatrix} j & k \\ \sqrt{2}u & 2v \end{vmatrix}$$

$$= 2\sqrt{2}\, v^2 i - 4u\, v j + 2\sqrt{2}\, u^2 k$$

$$d\vec{S} = \left(2\sqrt{2}\, v^2 i - 4uv j + 2\sqrt{2}\, u^2 k\right) du\, dv$$

$$dS = |d\vec{S}| = \left(4 \cdot 2 v^4 + 16 u^2 v^2 + 4 \cdot 2 \cdot u^4\right)^{1/2} du\, dv$$

$$\left(8v^4 + 16u^2v^2 + 8u^4\right)^{1/2}$$

$$\left[8\left(u^2 + v^2\right)^2\right]^{1/2}$$

$$dS = |d\vec{S}| = 2\sqrt{2}\left(u^2 + v^2\right) du\, dv$$

4.6 cont. p. 236

(3)

$$x = a\cos u$$
$$y = a\sin u$$
$$z = v$$

$$\frac{\partial R}{\partial u} = -a\sin u\, i + a\cos u\, j$$

$$\frac{\partial R}{\partial v} = k$$

$$\frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v} = \begin{vmatrix} i & j & k \\ -a\sin u & a\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = a\cos u\, i + a\sin u\, j$$

$$dS = \left(a^2\cos^2 u + a^2\sin^2 u\right)^{1/2} du\, dv = \underline{a\, du\, dv}$$

(4)

$$z = \underbrace{x^2 + y^2}_{f(x,y)} \qquad R = xi + yj + (x^2+y^2)k$$

See top p. 212

$$d\vec{S} = \frac{\partial R}{\partial x} \times \frac{\partial R}{\partial y}\, dx\, dy = \begin{vmatrix} i & j & k \\ 1 & 0 & 2x \\ 0 & 1 & 2y \end{vmatrix} dx\, dy$$

$$= (-2x\, i - 2y\, j + k)\, dx\, dy$$

$$dS = |d\vec{S}| = (4x^2 + 4y^2 + 1)^{1/2} dx\, dy.$$

4.6 cont. p. 236

(5) $x = u^2 \quad y = uv \quad z = \frac{1}{2}v^2$

$$d\vec{S} = \frac{\partial R}{\partial u} \times \frac{\partial R}{\partial v}\, du\, dv = \begin{vmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & v \end{vmatrix} du\, dv =$$

$$= i v^2 - 2uvj + 2u^2 k$$

$$|d\vec{S}| = \left(v^4 + 4u^2v^2 + 4u^4\right)^{1/2} du\, dv$$

$$\int_0^3\int_0^1 (v^4 + 4u^2v^2 + 4u^4)^{1/2}\, du\, dv$$

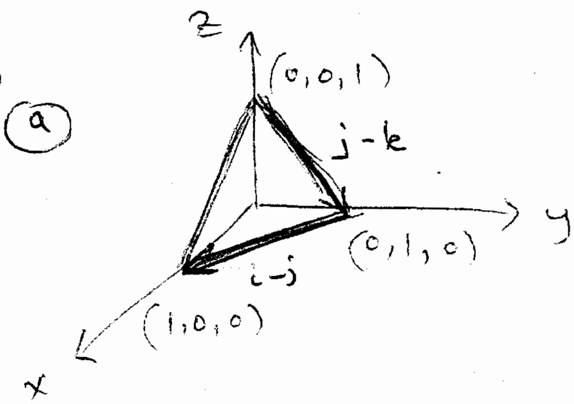
$$\int_0^3 \underbrace{\int_0^1 \left((2u^2+v^2)^2\right)^{1/2} du}\, dv$$

$$\int_0^3 \left(\frac{2}{3}u^3 + v^2 u\right)\Big|_0^1\, dv$$

$$= \int_0^3 \left(\frac{2}{3} + v^2\right) dv = \frac{2}{3}v + \frac{v^3}{3}\Big|_0^3 = 2 + 9 = \underline{\underline{11}}$$

4.6 cont. p.236

6 a



$$n = (j-k)\times(i-j)$$

$$= \begin{vmatrix} i & j & k \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$

$$= -i - j - k$$

vector perpendicular to the marked vectors (sides of triangle)

$\Rightarrow$ $\perp$ to third side.

unit $\Rightarrow$ $\dfrac{-i-j-k}{\sqrt{3}}$ this points toward origin (left hand)

$\Rightarrow$ requested vector $\dfrac{i+j+k}{\sqrt{3}}$

b. $\cos\gamma$ for this vector

γ angle of this vector with z

$$\cos\gamma = \frac{\dfrac{i+j+k}{\sqrt{3}}\cdot k}{1\cdot 1} = \frac{1}{\sqrt{3}}$$

lengths of each

c. $$\iint \frac{dx\,dy}{|\cos\gamma|} = \int_0^1 \int_0^{1-y} \frac{dx\,dy}{1/\sqrt{3}}$$

4.6 cont. p.236

6d.

$$\text{Integral} = \int_0^1 \sqrt{3}\, x \Big|_0^{1-y} dy = \int_0^1 \sqrt{3}\,(1-y)\, dy$$

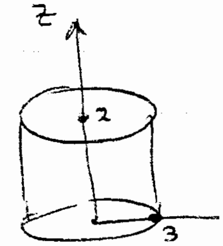
$$= \sqrt{3}\left(y - \frac{y^2}{2}\right)\Big|_0^1 = \sqrt{3}\left(1 - \frac{1}{2}\right) = \frac{\sqrt{3}}{2}$$

6e.

4.7 p. 246

(i) $F = z\vec{k}$

$F \cdot n = 0$ on the lateral surface



$F \cdot n = z$ on top $(n = \vec{k})$

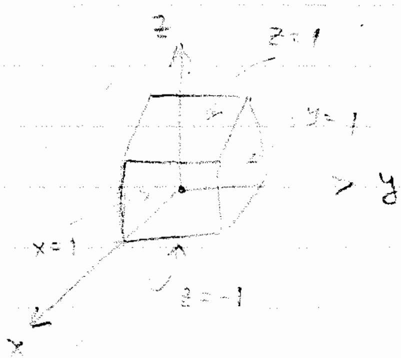
$F \cdot n = -z$ on bottom $(n = -\vec{k})$

$$\iint_{top} z\, dS - \iint_{bottom} z\, dS = 2 \underbrace{\iint_{top} dS}_{\pi r^2 = 9\pi} = \underline{\underline{18\pi}}$$

$z = 0$ on bottom

(2 d) $\iint (x^2 i + y^2 j + z^2 k) \cdot dS$

S surface of a cube $x = \pm 1$, $y = \pm 1$, $z = \pm 1$



$F \cdot n = (x^2 i + y^2 j + z^2 k) \cdot i$ if: $x = 1$

$= \qquad \cdot(-i) \qquad x = -1$

$= \qquad j \qquad y = 1$

$= \qquad (-j) \qquad y = -1$

$= \qquad (k) \qquad z = 1$

$= \qquad (-k) \qquad z = -1$

$F \cdot n = x^2 = 1$ (since $x = 1$)

area of square

$\iint dS = 2 \cdot 2 = 4$ on $x = 1$

$\iint -(-1)^2 dS = -2 \cdot 2 = -4$ on $x = -1$

thus these two faces yield a zero sum.

Similarly for other pairs

4.7 cont. p. 246

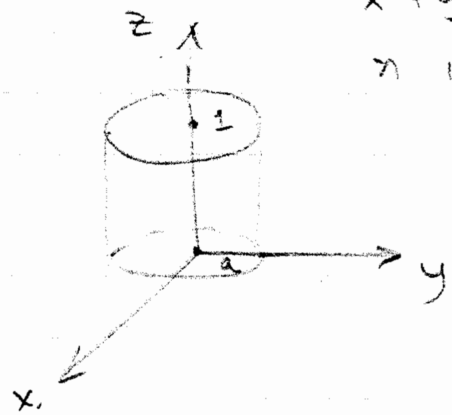
(5) $F = xi + yj + (z^2-1)k$

S: bounded by $z=0$

$z=1$

$x^2+y^2=a^2$ cylinder

n unit outward normal



on bottom $n = -k$

$$F \cdot n = -(z^2-1)\Big|_{z=0} = 1$$

$$\iint_{\text{bottom}} F \cdot n \, ds = \pi a^2 \quad \text{area of circle}$$

$$\iint_{\text{top}} F \cdot n \, ds = \iint (z^2-1)\Big|_{z=1} ds = 0$$

$$\iint_{\text{lateral}} F \cdot n \, ds = \iint_{\text{lateral}} (x^2+y^2)^{1/2} ds = a \underbrace{\iint ds}_{2\pi a \cdot 1} = 2\pi a^2$$

$$n = \frac{2xi + 2yj}{2\sqrt{x^2+y^2}}$$

$$\text{Total} = 2\pi a^2 + \pi a^2 = 3\pi a^2$$

4.7 p.246

(6) $F = yi + k$

box - no top

bottom $\quad n = -k \quad F\cdot n = -1 \quad \iint_S F\cdot n\, dS = -\iint_S dS$

$= -\underbrace{2\cdot 1}_{\text{area of rectangle}} = \underline{\underline{-2}}$

back : $\quad n = -i \quad F\cdot n = -j \quad \iint_S -y\, dS = -\int_0^1\int_0^1 y\, dy\, dz = \underline{\underline{-\frac{1}{2}}}$

right : $\quad n = j \quad F\cdot n = 0 \quad \iint_S F\cdot n\, dS = 0$

left : $\quad n = -j \quad F\cdot n = 0 \quad \iint_S F\cdot n\, dS = 0$

front : $n =$ outward normal to the slanted plane. Take 2 vectors on the plane

$(2,0,0)$ to $(2,1,0)$: j

$(2,1,0)$ to $(1,1,1)$: $-i+k$

$$n = \frac{j\times(-i+k)}{|j\times(-i+k)|} = \frac{\begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}}{\text{magnitude}} = \frac{i+k}{|i+k|} = \frac{i+k}{\sqrt{2}}$$

equation of that plane is $x+z = \text{Const.}$

Since the plane is thru $(2,1,0)$ $\underline{\underline{x+z=2}}$

Parametrize the plane $\quad x = u$
$\quad y = v$
$\quad z = 2-u$

$$\frac{\partial R}{\partial u} = i-k \qquad \frac{\partial R}{\partial v} = j \qquad \frac{\partial R}{\partial u}\times\frac{\partial R}{\partial v} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix} = i+k$$

$$dS = \left|\frac{\partial R}{\partial u}\times\frac{\partial R}{\partial v}\right| du\, dv = \sqrt{2}\, dx\, dy$$

4.7 cont. p.246

6 cont.

$F \cdot n$ on front $= (y\,i + k) \cdot \frac{i+k}{\sqrt{2}} = \frac{y+1}{\sqrt{2}}$

$$\iint_S F \cdot n\, dS = \int_0^1\int_1^2 \frac{y+1}{\cancel{\sqrt{2}}} \cancel{\sqrt{2}}\, dx\, dy = \int_0^1 x(y+1)\Big|_{x=1}^{x=2} dy$$

$$= \int_0^1 \left[2(y+1) - (y+1)\right] dy$$

$$= \int_0^1 (y+1)\, dy = \tfrac{1}{2}y^2 + y\Big|_0^1$$

$$= \tfrac{1}{2} + 1 = \underline{\underline{3/2}}$$

Total $= -2 - \frac{1}{2} + \frac{3}{2} = \underline{\underline{-1}}$

Remark:

If we want to use the divergence theorem we need D to be closed $\Rightarrow$

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_D \underbrace{\nabla \cdot F}_{=0}\, dV - \underbrace{\iint_{\text{top of box}} \vec{F} \cdot d\vec{S}}_{=1} = -1$$

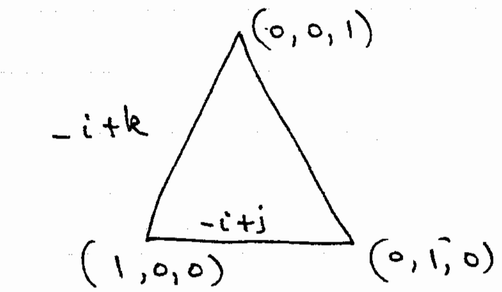
on top: $n = k$

$F \cdot n = 1$

$\iint_{top} 1\, \underbrace{dS}_{1} = 1$

4.7 cont. p. 246

(10) $\iint \vec{F} \cdot d\vec{S} := \iint \vec{F} \cdot \vec{n} \, dS$



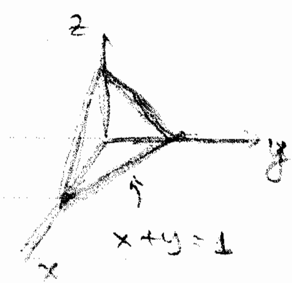
$$(-i+k)\times(-i+j) = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} = -i-j-k$$

$n = \dfrac{i+j+k}{\sqrt{3}}$ change sign for outward normal

$$F \cdot n = \frac{(x+1) - (2y+1) + z}{\sqrt{3}} = \frac{x-2y+z}{\sqrt{3}}$$

$$\cos\gamma = n \cdot k = \frac{1}{\sqrt{3}}$$

$$\iint F \cdot n \, dS = \int_0^1 \int_0^{1-x} \frac{(x-2y+\cancel{z}^{\,1-x-y})\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{3}}} \, dy \, dx$$



$$\int_0^1 \int_0^{1-x} (1-3y) \, dy \, dx$$

$x+y+z=1$

$z = 1-x-y$

$$= \int_0^1 \left(y - \frac{3}{2}y^2\right)\Big|_0^{1-x} dx = \int_0^1 1-x-\frac{3}{2}(1-x)^2 \, dx$$

$$= x - \frac{x^2}{2} + \frac{\cancel{3}}{2}\frac{(1-x)^3}{\cancel{3}}\Big|_0^1 = 1 - \frac{1}{2} - \frac{1}{2} = \underline{\underline{0}}$$

4.7 cont. p.246

(15) $E = -\nabla\left(|\vec{R}|^{-1}\right)$

a) show $E = \dfrac{R}{|\vec{R}|^3}$

$R = xi + yj + zk$

$|R| = \sqrt{x^2+y^2+z^2}$

$\dfrac{1}{|R|} = \left(x^2+y^2+z^2\right)^{-1/2}$

$$\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right)^{-1/2} = -\tfrac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2x) = -x\left(x^2+y^2+z^2\right)^{-3/2}$$

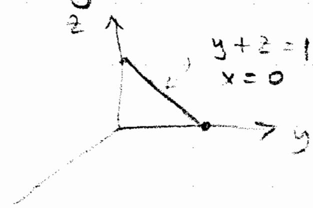
$$\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right)^{-1/2} = -\tfrac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y) = -y\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right)^{-1/2} = \qquad = -z\underbrace{\left(x^2+y^2+z^2\right)^{-3/2}}_{\frac{1}{|R|^3}}$$

$$-\nabla\left(|R|^{-1}\right) = \frac{xi+yj+zk}{|R|^3} = \frac{\vec{R}}{|\vec{R}|^3}$$

4.7 cont. p.246

15b $\int_C E \cdot dR$ C line segment from $(0,1,0)$ to $(0,0,1)$



$$\int_C E\cdot dR = \int \frac{R\cdot dR}{|R|^3}\Bigg|_{x=0} = \int_1^0 \frac{(y\,dy + (1-y)(-dy))}{[y^2+(1-y)^2]^{3/2}}$$

$$\int_1^0 \frac{(-1+2y)\,dy}{(1-2y+2y^2)^{3/2}} = \frac{1}{2}\int \frac{du}{u^{3/2}} = \frac{1}{2}(-2)u^{-1/2}$$

$u = 1-2y+2y^2$

$du = (-2+4y)\,dy$

$$= -u^{-1/2}$$

$$= (1-2y+2y^2)^{-1/2}\Big|_1^0$$

$$= 1-(1-2+2)^{-1/2} = \underline{\underline{0}}$$

c. $\iint_{S_1} E\cdot dS$ S_1 = Sphere $x^2+y^2+z^2=9$

$$\iint E\cdot n\,dS = \iint \frac{1}{|R|^2}\,dS = \frac{1}{9}\cdot 4\pi\cdot 3^2 = \underline{\underline{4\pi}}$$

$E = \dfrac{\vec{R}}{|\vec{R}|^3}$, $n = \dfrac{x\,i + y\,j + z\,k}{\sqrt{x^2+y^2+z^2}}$

$|\vec{R}| = \sqrt{x^2+y^2+z^2} = 3$ on sphere

4.7 cont. p. 246

(19) Hollow sphere $a \le r \le b$

Temperatures T_a, T_b

(a) Find the steady state temperature as a function of r.

As in the cylindrical example

$$Q\cdot n = -k\,\nabla T\cdot n \underset{\text{spheric. symm.}}{=} -k\frac{dT}{dr}$$

$$H = \iint_S Q\cdot n\,ds = -k\frac{dT}{dr}\underbrace{\iint_S ds}_{4\pi r^2} = -4\pi k r^2\frac{dT}{dr}$$

$$\int_a^b H\frac{dr}{r^2} = \int_{T_a}^{T_b} -4\pi k\,dT$$

$$-H\left.\frac{1}{r}\right|_a^b = -4\pi k(T_b - T_a)$$

$$H\left(\frac{1}{a}-\frac{1}{b}\right) = -4\pi k(T_b - T_a)$$

$$H = 4\pi k\frac{T_b - T_a}{\frac{1}{b}-\frac{1}{a}}$$

Substitute H in

$$H\int_a^r \frac{dr}{r^2} = -4\pi k\int_{T_a}^{T} dT$$

$$4\pi k\frac{T_b - T_a}{\frac{1}{b}-\frac{1}{a}}\left.\left(-\frac{1}{r}\right)\right|_a^r = -4\pi k(T - T_a)$$

$$\boxed{T = T_a + \frac{T_b - T_a}{\frac{1}{b}-\frac{1}{a}}\left(\frac{1}{r}-\frac{1}{a}\right)}$$

4.7 cont. p. 246

19b

r = halfway between a and b

$$= \frac{a+b}{2}$$

$$T_{\frac{a+b}{2}} = T_a + \frac{T_b - T_a}{\underbrace{\frac{1}{b} - \frac{1}{a}}_{\frac{a-b}{ab}}} \underbrace{\left(\frac{2}{a+b} - \frac{1}{a} \right)}_{\frac{\cancel{2}a - \cancel{a} - b}{a(a+b)} = \frac{a-b}{a(a+b)}}$$

$$= T_a + (T_b - T_a) \frac{\cancel{a}b}{\cancel{a-b}} \frac{\cancel{a-b}}{\cancel{a}(a+b)}$$

$$= T_a \underbrace{\left(1 - \frac{b}{a+b} \right)}_{\frac{a+\cancel{b}-\cancel{b}}{a+b}} + T_b \frac{b}{a+b}$$

$$= \frac{a T_a}{a+b} + \frac{b T_b}{a+b} \neq \frac{T_a + T_b}{2}$$

No

§ 4.8 p.256

2. The volume in example 4.28 is 3

$$\int_0^2\int_0^1\int_0^{1+x} 1\,dz\,dx\,dy = \int_0^2\int_0^1 z\Big|_0^{1+x} dx\,dy$$

$$= \int_0^2\int_0^1 (1+x)\,dx\,dy$$

$$= \int_0^2 x+\tfrac{1}{2}x^2\Big|_0^1 dy = \int_0^2 (1+\tfrac{1}{2})\,dy$$

$$= \tfrac{3}{2}y\Big|_0^2 = \underline{\underline{3}}$$

Let the reader try 3 other ways

$$\int_0^1\int_0^2\int_0^{1+x} dz\,dy\,dx$$

$$\int_0^1\int_0^{1+x}\int_0^2 dy\,dz\,dx$$

$$\int_0^1\int_0^2\int_0^1 dx\,dy\,dz + \int_1^2\int_0^2\int_{z-1}^1 dx\,dy\,dz$$

3. $$\int_0^2\int_0^3\int_0^{\sqrt{9-y^2}} dx\,dy\,dz$$

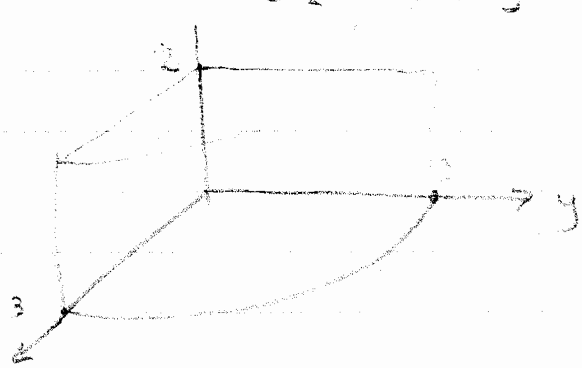
$0 \le x \le 9-y^2$

$x^2+y^2 \le 9$

<u>QUARTER</u> circle radius 3 center at origin

$0 \le y \le 3$

$0 \le z \le 2$



4.8 cont. 256

(4) a. $F = x^2 i + yj + zk$

$\iint F \cdot ds$ surface of cube $(0,1)\times(0,1)\times(0,1)$

Front	$n = i$		$F \cdot n = \underset{x=1}{x^2} = 1$
back	$n = -i$		$F \cdot n = \underset{x=0}{-x^2} = 0$
left	$n = -j$	$y = 0$	$F \cdot n = 0$
right	$n = j$	$y = 1$	$F \cdot n = y = 1$
bottom	$n = -k$	$z = 0$	$F \cdot n = 0$
top	$n = k$	$z = 1$	$F \cdot n = 1$

$$\underbrace{\iint_{\text{front}} ds}_{=1} + \underbrace{\iint_{\text{right}} ds}_{=1} + \underbrace{\iint_{\text{top}} ds}_{=1} = 3$$

unit squares

b. $f = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 2x + 1 + 1 = 2 + 2x$

$$\int_0^1\int_0^1\int_0^1 (2+2x)\,dx\,dy\,dz = \int_0^1\int_0^1 (2x + x^2)\Big|_0^1 dy\,dz = \int_0^1\int_0^1 3\,dy\,dz = 3$$

c. answers equal

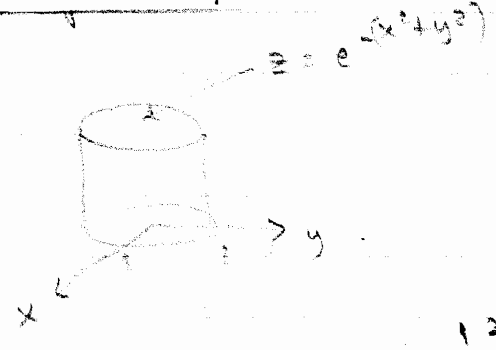
d. $F = x^2 i + y^2 j + z^2 k$

same again

e. $\iiint_V \nabla \cdot F \, dV = \iint_S F \cdot ds$ (Divergence theorem)

4.8 cont. p.256

(6)



$$\iiint dV = \int_0^1\int_0^{2\pi}\int_0^{e^{-r^2}} r\,dz\,d\theta\,dr$$

Note:
$r = \sqrt{x^2+y^2}$

$$= \int_0^1 \int_0^{2\pi} z \Big|_0^{e^{-r^2}} r\,dr\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} \left(e^{-r^2}\, r\,dr\right) d\theta$$

$$= 2\pi \int_0^1 r e^{-r^2}\,dr = \cancel{2}\pi \left(-\frac{1}{\cancel{2}} e^{-r^2}\right)\Big|_0^1$$

$$= -\pi\left(e^{-1} - 1\right)$$

$$= \pi\left(1 - \frac{1}{e}\right)$$

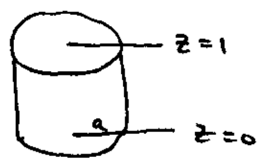
4.9 p. 262

3, 4, 6, 9

3. a $\quad F = x\,i - y\,j$ see 4.7 #4

$$\iint_S F\cdot n\,ds = \iiint_D \nabla\cdot F\,dV = \iiint_D 0\,dV = 0$$

$$\nabla\cdot F = 1 - 1 = 0$$



3b $\quad F = x\,i + y\,j + (z^2-1)\,k$

$$\nabla\cdot F = 1 + 1 + 2z$$

$$\int_0^{2\pi}\int_0^{a}\int_0^{1} (2+2z)\,dz\;r\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^{a} \underbrace{\left(z+\frac{z^2}{2}\right)\Big|_0^1}_{3/2}\; r\,dr\,d\theta$$

$$= 3\cdot \cancel{2}\pi\;\frac{1}{\cancel{2}}r^2\Big|_0^a = \underline{3\pi a^2}$$ 4.7 #5

4. $F = (x^2+xy)\,i + (y^2+yz)\,j + (z^2+zx)\,k$

$$\nabla \cdot F = \underline{2x}+y+2y+z+2z+\underline{x} = 3(x+y+z)$$

$$\int_{-1}^{1}\int_{-1}^{1}\underbrace{\int_{-1}^{1} 3(x+y+z)\,dx}\,dy\,dz$$

$$3\left(\frac{x^2}{2}+xy+xz\right)\Big|_{x=-1}^{x=1}$$

$$3\left(\cancel{\tfrac{1}{2}}+y+z-\cancel{\tfrac{1}{2}}+y+z\right)$$

$$\int_{-1}^{1}\underbrace{\int_{-1}^{1} 6(y+z)\,dy}\,dz$$

$$6\left(\frac{y^2}{2}+yz\right)\Big|_{y=-1}^{y=1}$$

$$6\left(\tfrac{1}{2}+z-\tfrac{1}{2}+z\right)$$

$$12z$$

$$= \int_{-1}^{1} 12z\,dz = 6z^2\Big|_{-1}^{1} = 0$$

6. a. $$\vec{F} = F_r(r)\, e_r$$

$$\nabla \cdot F = r^m \qquad m \geq 0$$

$$\nabla \cdot F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r(r)\right) = r^m$$

$$\frac{\partial}{\partial r}\left(r^2 F_r(r)\right) = r^{m+2}$$

$$r^2 F_r(r) = \frac{r^{m+3}}{m+3} \qquad \text{(never mind the const.)}$$

$$F_r(r) = \frac{r^{m+1}}{m+3}$$

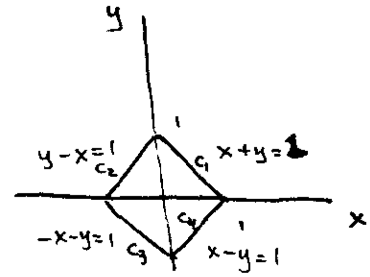
b. $$\iiint_D \underbrace{r^m}_{\nabla \cdot F}\, dV = \iiint_D \nabla \cdot F\, dV = \iint_S \vec{F} \cdot \vec{n}\, ds$$

$$= \iint_S \frac{r^{m+1}}{m+3}\, e_r \cdot dS = \frac{1}{m+3}\iint_S r^{m+1}\, e_r \cdot ds$$

9. a $\quad F = x\,i$

$$\iint_S \nabla \times F \cdot n \, ds = \int_C F \cdot dR$$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = 0$$



$$\iint_S \nabla \times F \cdot n \, ds = 0$$

$$\int_C F \cdot dR = \int_{C_1} x\,dx + \int_{C_2} x\,dx + \int_{C_3} x\,dx + \int_{C_4} x\,dx$$

$$= \int_1^0 x\,dx + \int_0^{-1} x\,dx$$

$$= \tfrac{1}{2}x^2 \Big|_1^0 + \tfrac{1}{2}x^2 \Big|_0^{-1} + \tfrac{1}{2}x^2 \Big|_{-1}^0 + \tfrac{1}{2}x^2 \Big|_0^1$$

$$= 0 - \tfrac{1}{2} + \tfrac{1}{2} - 0 \quad + 0 - \tfrac{1}{2} + \tfrac{1}{2} - 0$$

$$= 0 + 0 = 0$$

b. $\quad F = -y\,i + x\,j$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2k \qquad n = k \text{ (square is in xy plane)}$$

$$\iint_S \nabla \times F \cdot n \, ds = 2 \underbrace{\iint_S ds}_{(\sqrt{2})^2} = 2 \cdot 2 = \underline{\underline{4}}$$

$$\int_C F \cdot dR = \int_{C_1 + C_2 + C_3 + C_4} (-y\,dx + x\,dy) \Rightarrow \int_1^0 \left[(\cancel{x} - 1)\,dx - \cancel{x}\,dx\right] = -x \Big|_1^0 = \underline{\underline{1}} = \int_{C_1} F \cdot dR$$

$y = 1 - x$

$dy = -dx$

$$\int_{C_2} F \cdot dR = \int_0^{-1} \left[(-\cancel{x}-1)\,dx + \cancel{x}\,dx \right] = -x \Big|_0^{-1} = \underline{\underline{1}}$$

~~x = y+1~~

$y = x+1 \qquad dy = dx$

$$\int_{C_3} F \cdot dR = \int_{-1}^{0} \left[(1+\cancel{x})\,dx - \cancel{x\,dx} \right] = x \Big|_{-1}^{0} = \underline{\underline{1}}$$

$y = -1 - x$

$dy = -dx$

$$\int_{C_4} F \cdot dR = \int_0^{1} \left[(1-\cancel{x})\,dx + \cancel{x}\,dx \right] = x \Big|_0^{1} = \underline{\underline{1}}$$

$y = x-1$

$dy = dx$

$$\int_{C} F \cdot dR = 1+1+1+1 = \underline{\underline{4}}$$

14. Use Stokes'

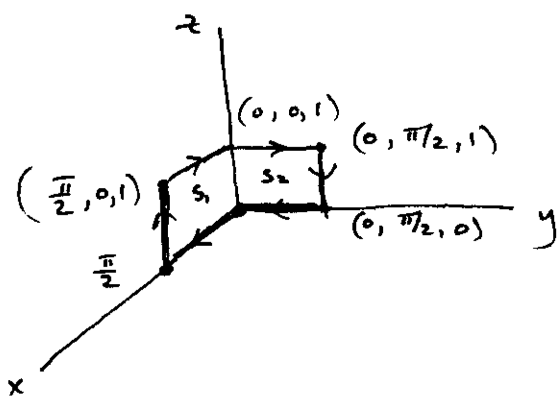
$$\int_C \underbrace{(x\sin y\, i - y\sin x\, j + (x+y)z^2 k)}_{\vec{F}} \cdot dR$$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x\sin y & -y\sin x & (x+y)z^2 \end{vmatrix}$$

$$= i\,(z^2+0) - j\,(z^2-0) + k\,(-y\cos x - x\cos y)$$

$$= z^2 i - z^2 j - (x\cos y + y\cos x)k$$

$$\iint_S \nabla\times F\cdot n\, dS$$



S is made of
2 pieces one in xz plane and one in yz plane.
For the first $\vec{n}_1 = j$ and for the second $n_2 = i$

$$\nabla\times F\cdot n_1 = -z^2$$
$$\nabla\times F\cdot n_2 = z^2$$

$$\iint_{S_1} -z^2\, dS + \iint_{S_2} z^2\, dS$$

S_1 : $y=0$ $\quad 0\le x\le \pi/2$, $0\le z\le 1$

x, z are parameters

$y = 0.x + 0.z$

$$\iint_{S_1} (-z^2)\, dS = \int_0^{\pi/2} \underbrace{\int_0^1 (-z^2)\, dz}_{-\frac{z^3}{3}\Big|_0^1 = -\frac{1}{3}}\, dx = -\frac{1}{3} \int_0^{\pi/2} dx = \underline{-\frac{1}{3}\cdot\frac{\pi}{2}}$$

$$\iint_{S_2} z^2\, dS = \int_0^{\pi/2} \underbrace{\int_0^1 z^2\, dz}_{\frac{z^3}{3}\Big|_0^1 = \frac{1}{3}}\, dy = \underline{\frac{1}{3}\cdot\frac{\pi}{2}}$$

$$\Rightarrow \iint_S \nabla \times F \cdot n\, dS = -\frac{1}{3}\cdot\frac{\pi}{2} + \frac{1}{3}\cdot\frac{\pi}{2} = \underline{\underline{0}}$$

#17 $\varphi(x,y,z) = xyz + 5$

$$\iint_S \nabla\varphi \cdot n \; ds$$

$x^2+y^2+z^2=9$

$\nabla\varphi = yz\,i + xz\,j + xy\,k$

$$\vec{n} = \frac{2xi+2yj+2zk}{\sqrt{4x^2+4y^2+4z^2}} = \frac{\cancel{2}xi+\cancel{2}yj+\cancel{2}zk}{\cancel{2}\cdot 3}$$

$$\nabla\varphi \cdot n = \frac{\cancel{3}xyz}{\cancel{3}}$$

$$\int_0^{2\pi}\int_0^{\pi} \underbrace{3\sin\varphi\cos\theta}_{x} \cdot \underbrace{3\sin\varphi\sin\theta}_{y} \cdot \underbrace{3\cos\varphi}_{z} \cdot \underbrace{9\sin\varphi\, d\varphi\, d\theta}_{ds}$$

$$= 3^5 \int_0^{2\pi}\int_0^{\pi} \sin^3\varphi \quad \cos\varphi\, d\varphi \quad \sin\theta\cos\theta\, d\theta \quad = 0$$

$u = \sin\varphi$

$du = \cos\varphi\, d\varphi$

$$\frac{u^4}{4} = \frac{\sin^4\varphi}{4}\Big|_0^{\pi} = 0$$

4.10

1. $S_t = x^2+y^2+z^2 = (vt)^2 \qquad z \geq 0$

$\vec{F}(\vec{R},t) = R\,t$

$$\Phi(t) = \iint_{S_t} F(R,t)\cdot dS$$

$$\frac{d\Phi(t)}{dt} = \iint_{S_t} \Big(\underbrace{\frac{\partial F}{\partial t}}_{\vec{R}} + \underbrace{\nabla\cdot F}_{3t}\,\vec{v}\Big)\cdot dS + \oint_{C_t} \underbrace{F\times v}_{R\times v\,t}\cdot dR$$

$$\nabla\cdot F(R,t) = \qquad 3t$$

$$\Phi(t) = \int_0^{2\pi}\int_0^{\pi} \left(vt^2\sin\varphi\cos\theta\, i + vt^2\sin\varphi\sin\theta\, j + vt^2\cos\varphi\, k\right)\cdot\left(\frac{\partial R}{\partial\varphi}\times\frac{\partial R}{\partial\theta}\right) d\varphi\, d\theta$$

spherical
$$\begin{cases} x = vt\sin\varphi\cos\theta \\ y = vt\sin\varphi\sin\theta \\ z = vt\cos\varphi \end{cases}$$

$$\frac{\partial R}{\partial\varphi} = vt\cos\varphi\cos\theta\, i + vt\cos\varphi\sin\theta\, j - vt\sin\varphi\, k$$

$$\frac{\partial R}{\partial\theta} = -vt\sin\varphi\sin\theta\, i + vt\sin\varphi\cos\theta\, j$$

$$\frac{\partial R}{\partial\varphi}\times\frac{\partial R}{\partial\theta} = v^2t^2\sin^2\varphi\cos\theta\, i + v^2t^2\sin^2\varphi\sin\theta\, j + v^2t^2\sin\varphi\cos\varphi\, k$$

$$\text{Integrand} = v^3t^4 \underbrace{\Big(\underbrace{\sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta}_{\sin^3\varphi} + \sin\varphi\cos^2\varphi\Big)}_{\sin\varphi(\sin^2\varphi+\cos^2\varphi)} d\varphi\, d\theta$$

$$= \sin\varphi$$

$$\Phi(t) = \int_0^{2\pi}\int_0^{\pi/2} \sin\varphi \quad v^3t^4\, d\varphi\, d\theta = v^3t^4\, 2\pi\, \underbrace{(-\cos\varphi)\Big|_0^{\pi/2}}_{1} = \underline{\underline{2\pi v^3 t^4}}$$

$$\boxed{\frac{d\Phi}{dt} = 8\pi v^3 t^3}$$

$$\iint_{S_t} (\vec{R} + 3tv) \cdot ds =$$

$$\int_0^{2\pi} \int_0^{\pi/2} (\underbrace{vt \sin\varphi\cos\theta i + vt \sin\varphi \sin\theta j + vt^2 \cos\varphi k + 3\vec{v} t}_{\vec{R} = \frac{\partial F}{\partial t}}) \cdot \underbrace{\frac{\partial R}{\partial \varphi} \times \frac{\partial R}{\partial \theta}}\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \left[v^3 t^3 \sin\varphi + 3t \underbrace{\quad \frac{\partial \vec{R}}{\partial t}}_{= R} \cdot \left(\frac{\partial R}{\partial \varphi} \times \frac{\partial R}{\partial \theta} \right) \right] d\varphi\, d\theta$$

$$= 4 v^3 t^3 \int_0^{2\pi} \int_0^{\pi} \sin\varphi \; d\varphi\, d\theta = \underline{8 v^3 t^3 \pi}$$

$$\oint_{C_t} (R \times \vec{v}) t \cdot dR = 0$$

$$\downarrow$$

$$\underbrace{\frac{dR}{dt}}_{=0}$$

$$\underbrace{R \times \frac{dR}{dt} \cdot \frac{dR}{dt}}_{=0} t\, dt$$

$$\Rightarrow \quad 8\pi v^3 t^3 = \frac{d\Phi}{dt} = 8 v^3 t^3 \pi + 0 \quad ✓$$

5.1 p.276

6 $F = x^3 i + yx j - x^3 k$

$\operatorname{div} F = 3x^2 + x$

at $(3,1,-2)$ $\operatorname{div} F = 27 + 3 = \underline{30}$

7 $F = 3x i + y j + z k$

$\operatorname{div} F = 3 + 1 + 1 = \underline{\underline{5}}$ flux per unit volume

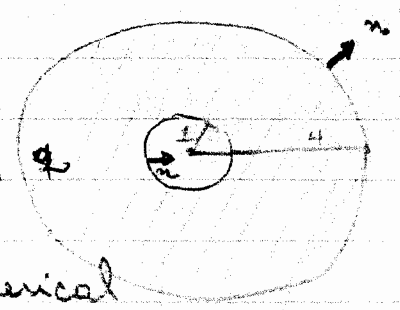
$\Rightarrow$ flux $= 5v$

8 $F = 3x^2 i + y j + z k$

$\operatorname{div} F = 6x + 1 + 1$

flux depends on x $\Rightarrow$ location also

9 a



Sphere of radius 4 with a hole in the center of spherical shape & radius 1.

The normal is in the direction of r or -r

b use spherical coordinates

For the outer sphere $n = r$

inner $n = -r$

c They are equal. The only source is in the origin and flow inside (n pointing inside) and out the outer sphere. No other source to influence the outcome.

d. No difference.

5.1 cont.

9e

$$\iint_S \frac{xi+yj+zk}{(x^2+y^2+z^2)^{3/2}} \cdot n\, ds$$

S : ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$

$$\frac{\partial F_1}{\partial x} = \frac{1\cdot(x^2+y^2+z^2)^{3/2} - \left[\frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot 2x\right]x}{\left[(x^2+y^2+z^2)^{3/2}\right]^2}$$

$$= \frac{(x^2+y^2+z^2)^{3/2} - 3x^2(x^2+y^2+z^2)^{1/2}}{(x^2+y^2+z^2)^3}$$

$$= \frac{x^2+y^2+z^2-3x^2}{(x^2+y^2+z^2)^{5/2}}$$

By symmetry

$$\frac{\partial F_2}{\partial y} = \frac{x^2+y^2+z^2-3y^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\frac{\partial F_3}{\partial z} = \frac{x^2+y^2+z^2-3z^2}{(x^2+y^2+z^2)^{5/2}}$$

$$\Rightarrow \frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z} = \frac{3(x^2+y^2+z^2)-3x^2-3y^2-3z^2}{(x^2+y^2+z^2)^{5/2}} = 0$$

$\Rightarrow \nabla\cdot F = 0$

But F is not continuous at the origin, so we cut the origin out (taking a small sphere centered at the origin.)

$$\iint_S \frac{xi+yj+zk}{(x^2+y^2+z^2)^{3/2}} \cdot n\, ds + \iint_{\substack{\text{Ball} \\ \text{centered} \\ \text{at origin}}} F \cdot n\, ds = \underbrace{\iiint_D \underbrace{\nabla \cdot F}_{=0}\, dV}_{=0}$$

$$\Rightarrow \iint_{\text{Ball}} F \cdot n\, ds = ?$$

$$n = -\frac{xi+yj+zk}{\sqrt{x^2+y^2+z^2}}$$ (Pointing outside, i.e. toward origin)

$$F \cdot n = -\frac{1}{x^2+y^2+z^2} = -\frac{1}{r^2}$$

Using spherical coordinates

$$\iint_{\text{Ball}} F \cdot n\, ds = -\int_0^{\pi}\int_0^{2\pi} \frac{1}{r^2} r^2 \sin\varphi \, d\theta\, d\varphi = -2\pi(-\cos\varphi)\Big|_{\varphi=0}^{\varphi=\pi}$$

$$= -2\pi(1+1) = 4\pi$$

$$\Rightarrow \iint_S F \cdot n\, ds = 4\pi$$

1.

$$\varphi = xyz + 5$$

$$\nabla\varphi = yz\,i + xz\,j + xy\,k$$

$$\iint_S \left(\frac{1}{R}\nabla\varphi - \varphi\nabla\frac{1}{R}\right)\cdot d\vec{S} = \iint_S \left(\frac{1}{R}\nabla\varphi - \varphi\nabla\frac{1}{R}\right)\cdot n\,dS$$

$$\underset{(4.54)}{=} 4\pi\underbrace{\varphi(0,0,0)}_{=5} + \iiint_D \frac{\nabla^2\varphi}{R}\,dV$$

$$\nabla^2\varphi = 0$$

$$= \underline{20\pi}$$

2a

$$\varphi = x^2 + y^2 - 2z^2 + 4$$

$$\nabla^2\varphi = 2 + 2 - 4 = 0$$

$$\varphi(0,0,0) = 4$$

$$\underline{-16\pi}$$

$\varphi(0,0,0) = -\frac{1}{4\pi}\iint_S$ Integrand has a minus in front!

2b

$$\varphi = x^2 - z^2 + 5$$

$$\nabla^2\varphi = 2 - 2 = 0$$

$$\varphi(0,0,0) = 5$$

$$\underline{-20\pi}$$

5.2 cont. p.284

8a. Prove $\displaystyle\iiint_{D} \nabla\varphi \cdot \nabla\times F \, dV = \iint_{S} F\times\nabla\varphi \cdot dS$

Using (3.36) $\underbrace{G}_{\equiv\nabla\varphi} \cdot \nabla\times F = F \cdot \underbrace{\nabla\times G}_{\underbrace{\nabla\times\nabla\varphi}_{=0 \text{ by } 3.40}} + \nabla\cdot(F\times G)$

$$\Rightarrow \iiint_{D} \nabla\varphi \cdot \nabla\times F \, dV = \iiint_{D} \nabla\cdot(F\times \underbrace{G}_{\nabla\varphi}) \, dV$$

$$= \iint_{S} F\times\nabla\varphi \cdot dS$$

5.4 p.294

4. Derive (5.49)

$$\int_C \tfrac{1}{2}(x\,dy - y\,dx) = A$$

Let $F = \underbrace{-\tfrac{1}{2}y}_{F_1}\ i + \underbrace{\tfrac{1}{2}x}_{F_2}\ j$

$$\Rightarrow \frac{\partial F_1}{\partial y} = -\frac{1}{2} \qquad \frac{\partial F_2}{\partial x} = \frac{1}{2}$$

$$\int_C (F_1\,dx + F_2\,dy) = \int_C \left(-\tfrac{1}{2}y\,dx + \tfrac{1}{2}x\,dy\right)$$

$$= \tfrac{1}{2}\int_C (x\,dy - y\,dx)$$

On the other hand

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx\,dy = \iint_D \left(\tfrac{1}{2} + \tfrac{1}{2}\right)dx\,dy = \iint_D dx\,dy = A$$

5. $R = x\,i + y\,j$

$dR = dx\ i + dy\ j$

a. $|\vec{R} \times (\vec{R} + d\vec{R})| = ?$ $\left(\vec{R} \times (\vec{R} + d\vec{R})\right) = \begin{vmatrix} i & j & k \\ x & y & 0 \\ x+dx & y+dy & 0 \end{vmatrix} = k\left[x(\cancel{y}+dy) - y(\cancel{x}+dx)\right]$

$$= k(x\,dy - y\,dx)$$

and the magnitude is $|x\,dy - y\,dx|$

b. $$\int_C \tfrac{1}{2}(x\,dy - y\,dx) = \int_C \underbrace{\tfrac{1}{2}\underbrace{|\vec{R} \times (\vec{R} + d\vec{R})|}_{\text{area of parallelogram}}}_{\text{area of triangle}}$$

5.4 cont. p. 294

5c.

7. $C: x^2+y^2=9$

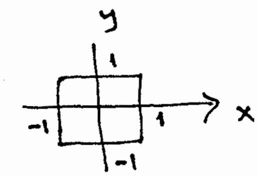
$F = yi - 3xj$

$$\int_C (y\,dx - 3x\,dy) = \iint_D (-3-1)\,dx\,dy = -4\int_0^{2\pi}\int_0^3 r\,dr\,d\theta$$

$$= -4 \cdot \cancel{2}\pi \cdot \frac{1}{\cancel{2}} r^2 \Big|_0^3 = -4\pi \cdot 9 = -36\pi$$

5.4 cont. p.294

9. $\int_C \left(4y^3\,dx - 2x^2\,dy\right) = \iint_D \left(-4x - 12y^2\right)\,dx\,dy$



a. $y = -1 \quad -1 \le x \le 1 \qquad \int_{-1}^{1} -4\,dx = -4x\Big|_{-1}^{1} = \underline{-8}$

$x=1 \qquad \int_{-1}^{1} -2\,dy = \underline{-4} \quad ; \quad y=1 \qquad \int_{1}^{-1} 4\,dx = 4x\Big|_{1}^{-1} = \underline{-8}$

$x=-1 \qquad \int_{1}^{-1} -2\,dy = -2y\Big|_{1}^{-1} = \underline{4} \qquad \Rightarrow$ Total $\underline{\underline{-16}}$

contributions of second term cancel !!

b. using Green's theorem

$$\iint_D \left(-4x - 12y^2\right)\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1} \left(-4x - 12y^2\right)\,dx\,dy$$

$$= \int_{-1}^{1} -\cancel{4}^{\,2}\,\frac{x^2}{\cancel{2}} - 12xy^2\Big|_{-1}^{1}\,dy = \int_{-1}^{1}\left(\cancel{-2} - 12y^2 - (\cancel{-2} + 12y^2)\right)dy$$

$$= -24\int_{-1}^{1} y^2\,dy = -24\,\frac{y^3}{3}\Big|_{-1}^{1} = -24\left(\frac{1}{3} + \frac{1}{3}\right) = \underline{\underline{-16}}$$

Same as a.

5.4 Cont. p. 294

9 c symmetry

As we noted in a the contribution of

$$\int_C -2x^2 dy$$

cancel and we are left with

$$\int_C 4y^3 dx$$

The integral on $y=1$ goes in the opposite direction to integral on $y=-1$ $\Rightarrow$ add up

$$4\int_C (-1)^3 dx = -4\int_{-1}^{1} dx = -8 \text{ each}$$

Total $\underline{\underline{-16}}$

---

12. Use Green's theorem to find the area inside

$$x = \frac{t}{1+t^3}, \quad y = \frac{t^2}{1+t^3} \qquad 0 \le t < \infty$$

$$A = \frac{1}{2}\int_C (x\,dy - y\,dx) = \frac{1}{2}\int_0^\infty \left(\frac{t}{1+t^3} \cdot \frac{2t(1+t^3) - 3t^2 \cdot t^2}{(1+t^3)^2} - \frac{t^2}{1+t^3} \frac{1+t^3 - 3t^2 \cdot t}{(1+t^3)^2} \right)$$

$$= \frac{1}{2}\int_0^\infty \frac{t[2t - t^4] - t^2(1-2t^3)}{(1+t^3)^3} dt = \frac{1}{2}\int_0^\infty \frac{\cancel{2}t^2 - \cancel{t^5} - \cancel{t^2} + \cancel{2}t^5}{(1+t^3)^3} dt$$

$$= \frac{1}{2}\int_0^\infty \frac{t^2\cancel{(1+t^3)}}{(1+t^3)^{\cancel{3}\,2}} dt = \frac{1}{2}\cdot\frac{1}{3}\int \frac{du}{u^2} = -\frac{1}{6u} = -\frac{1}{6(1+t^3)}\Big|_0^\infty$$

$u = 1+t^3$

$du = 3t^2 dt$

$$= 0 + \frac{1}{6}$$

5.5 p. 299

1.

$F = 3y\,i + (5-2x)\,j + (z^2-2)\,k$

a. $\nabla \cdot F = 0 + 0 + \underline{\underline{2z}}$

b. $\text{curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 5-2x & z^2-2 \end{vmatrix} = i \cdot 0 - j \cdot 0 + k(-2-3) = \underline{\underline{-5k}}$

c. $\iint_S \text{curl } F \cdot n\, dS$

$$0 = \iiint_V \underbrace{\nabla \cdot (\nabla \times F)}_{=0}\, dV = \iint_S \text{curl } F \cdot n\, dS + \iint_{\text{Disk}} \text{curl } F \cdot n\, dS$$

$$\Rightarrow \iint_S \text{curl } F \cdot n\, dS = -\iint_{\text{Disk}} -5k \cdot \underset{\uparrow \atop -k}{n}\, dS = -5 \underbrace{\iint_{\text{disk}} dS}_{4\pi}$$

$= \underline{-20\pi}$

5.5 cont p. 299

2. $\nabla \times F = 2y\,i - 2z\,j + 3k$

a.

$$\iiint_V \nabla \cdot (\nabla \times F)\, dV = 0$$

$$= \iint_S \nabla \times F \cdot n\, dS + \iint_{Disk} \nabla \times F \cdot n\, dS$$

$$\Rightarrow \iint_S (\nabla \times F) \cdot n\, dS = -\iint_{Disk} (\nabla \times F) \cdot \underbrace{n}_{-k}\, dS$$

$$= -\iint_{Disk} -3\, dS = 3(\pi \cdot 9) = \underline{-27\pi}$$

b. $$\iiint_V \nabla \cdot (\nabla \times F)\, dV = 0$$

$$= \iint_{\substack{S \\ \text{whole} \\ \text{Sphere}}} (\nabla \times F) \cdot n\, dS$$

$$\Rightarrow \iint_S \nabla \times F \cdot n\, dS = 0$$

5.5 cont. p. 299

3 $$\iint_S \nabla\varphi \times \nabla\psi \cdot ds = \iint_S \nabla \times (\varphi \nabla \psi) \cdot ds =$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \end{vmatrix} = i\left(\underbrace{\frac{\partial \varphi}{\partial y}\frac{\partial \psi}{\partial z} - \frac{\partial \varphi}{\partial z}\frac{\partial \psi}{\partial y}}\right)$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi\frac{\partial \psi}{\partial x} & \varphi\frac{\partial \psi}{\partial y} & \varphi\frac{\partial \psi}{\partial z} \end{vmatrix} = i\left(\frac{\partial\left(\varphi\frac{\partial \psi}{\partial z}\right)}{\partial y} - \frac{\partial\left(\varphi\frac{\partial \psi}{\partial y}\right)}{\partial z}\right)$$

$$\frac{\partial \varphi}{\partial y}\frac{\partial \psi}{\partial z} + \cancel{\varphi\frac{\partial^2 \psi}{\partial z \partial y}} - \frac{\partial \varphi}{\partial z}\frac{\partial \psi}{\partial y} - \cancel{\varphi\frac{\partial^2 \psi}{\partial y \partial z}}$$

as above

similarly for other components.

$$= \int_C \varphi \nabla \psi \cdot dR$$

by stokes Theorem

5.5 cont. p.299

7. $F = \nabla \varphi$

$$\int_C F \cdot dR = \varphi(b) - \varphi(a) = 0 \text{ for closed curves}$$

a. $\iint_S \nabla \times \nabla \varphi \cdot n \, dS = \int_C \nabla \varphi \cdot dR = 0$

b. $n \cdot \nabla \times (\nabla \varphi) = 0$ because of a

c. $\Rightarrow \nabla \times \nabla \varphi = \vec{0}$

d. The identity in Section 3.8 is $\nabla \times \nabla \varphi = 0$